Homework # 6 Solutions

set of units used: MKSA

-Problem 1- Find all elements of the Maxwell stress tensor for a monochromatic plane wave traveling in the $\hat{\mathbf{x}}$-direction and linearly polarized in the $\hat{\mathbf{y}}-$direction. Does your answer make sense? (Remember that $\mathbf{T}$ represent the momentum flux density.) How is the momentum flux density related to the energy density?

SOLUTION

Monochromatic plane wave traveling in the $\hat{\mathbf{x}}$-direction and linearly polarized in the $\hat{\mathbf{y}}-$direction

$$\left\{ \begin{array}{l} \mathbf{E}(x,t) = E_o \cos(kx - \omega t + \delta)\hat{\mathbf{y}} \quad , \\ \mathbf{B}(x,t) = B_o \cos(kx - \omega t + \delta)\hat{\mathbf{z}} \quad . \end{array} \right. \tag{1}$$

Element (i,j) of the Maxwell stress tensor

$$T_{i,j} = \varepsilon_o(E_iE_j - \frac{1}{2}\delta_{i,j}E^2) + \frac{1}{\mu_o}(B_iB_j - \frac{1}{2}\delta_{i,j}B^2) \quad . \tag{2}$$

Calculating $T_{i,j}$ for $i,j = 1,2,3$ using the fields in eqns. (1) one gets

$$\mathbf{T} = \begin{pmatrix} -\varepsilon_o|\mathbf{E}(x,t)|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

The result (3) make sense because $-T_{i,j}$ is the momentum carried by the fields (1) on the $\hat{\mathbf{x}}-$direction, crossing a surface oriented in the $\hat{\mathbf{y}}-$direction per unit area per unit time. Since the fields (1) represents a wave traveling in the $\hat{\mathbf{x}}-$direction then the flow of momentum density will be different from zero only across a surface oriented in the $\hat{\mathbf{x}}-$direction. This means that only the first column of the matrix $\mathbf{T}$ can have elements different from zero. Result (3) shows how only the first element of such column is effectively different from zero. This is due to the electromagnetic plane wave property of carrying a momentum parallel to the direction of propagation ($\hat{\mathbf{x}}$ in our case, $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$, where $\mathbf{k}$ is the wave vector, in general) [1].

The momentum flux density as we have already seen is defined by the matrix $-\mathbf{T}$. The energy density is defined by the scalar $W \equiv (\varepsilon_o E^2 + B^2/\mu_o)/2 = \varepsilon_o E^2$ $(= SS \cdot \hat{\mathbf{k}}/v)$, where $\mathbf{S}$ is the Poynting vector and $v = c/n$ the phase velocity of the wave (c is the speed of light and n the index of refraction of the material).
Since in a plane wave $\mathbf{E}$, $\mathbf{B}$ and $\hat{\mathbf{k}}$ are an orthogonal ordered set of vectors

$$\mathbf{E} \times \mathbf{B} = \hat{\mathbf{k}}\frac{E^2}{v} \quad , \tag{4}$$

then if we define the vector $\mathbf{v}$ as $v\hat{\mathbf{k}}$ we get

$$\sum_{j=1}^{3} T_{i,j} v_j = \sum_{j=1}^{3} [\varepsilon_o(E_i E_j - \frac{1}{2}\delta_{i,j}E^2) + \frac{1}{\mu_o}(B_i B_j - \frac{1}{2}\delta_{i,j}B^2)]v_j \tag{5}$$

$$= -\frac{1}{2}v_i[\varepsilon_o E^2 + \frac{1}{\mu_o}B^2] = -v_i W \quad , \tag{6}$$

where we have used the properties $\mathbf{E} \cdot \mathbf{v} = \mathbf{B} \cdot \mathbf{v} = 0$. Result (5) can be rewritten more concisely as

$$-\mathbf{T}_i \cdot \mathbf{v} = v_i W (= S_i) \quad , \qquad \forall \;\; i = 1, 2, 3 \tag{7}$$

[1] See the effect of the *pressure of light*.

$\boxed{\text{-Problem 2-}}$ Prove that in the problem of normal incidence of an electromagnetic plane wave on the boundary between two linear media, the reflected and transmitted wave must have the same polarization of the incident wave. (Let the polarizations of the transmitted and reflected wave be $\hat{\mathbf{n}}_T = \cos\theta_T \hat{\mathbf{y}} + \sin\theta_T \hat{\mathbf{z}}$, and $\hat{\mathbf{n}}_R = \cos\theta_R \hat{\mathbf{y}} + \sin\theta_R \hat{\mathbf{z}}$ respectively. Then prove from the boundary conditions that $\theta_T = \theta_R = 0$.)

SOLUTION

Suppose the yz plane forms the boundary between two linear media. A plane wave of frequency ω, traveling in the $\hat{\mathbf{x}}-$direction and polarized in the $\hat{\mathbf{y}}-$direction, approaches the interface from the left (see figure 1).

The electric and magnetic fields of the incident (I) wave can be written as

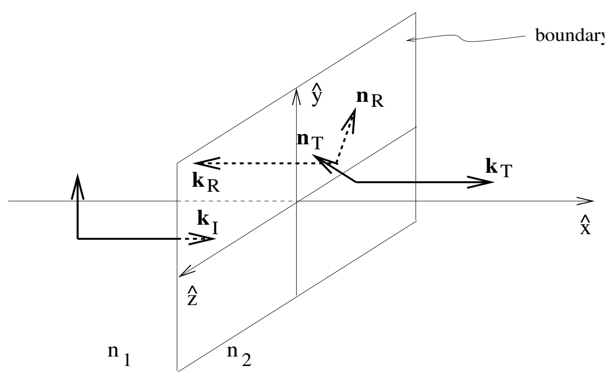

Figure 1: Reflection and transmission at normal incidence, without any apriori assumption on the polarization of the reflected and of the transmitted wave.

follows

$$\begin{cases} \mathbf{E}_I(x,t) = E_{oI} e^{i[k_I x - \omega t]} \hat{\mathbf{y}} \quad , \\ \mathbf{B}_I(x,t) = \hat{\mathbf{x}} \times \dfrac{\mathbf{E}_I}{v_1} \quad . \end{cases} \tag{1}$$

It gives rise to a transmitted (T) and a reflected (R) wave

$$\begin{cases} \mathbf{E}_{R,T}(x,t) = E_{oR,T} e^{i[(-)^{\alpha_{R,T}} k_{R,T} x - \omega t]} \hat{\mathbf{n}}_{R,T} \quad , \\ \mathbf{B}_{R,T}(x,t) = \hat{\mathbf{k}}_{R,T} \times \dfrac{\mathbf{E}_{R,T}}{v_{1,2}} \quad , \end{cases} \tag{2}$$

where $\alpha_T = -\alpha_R = 1$ and $\hat{\mathbf{k}}_T = -\hat{\mathbf{k}}_R = \hat{\mathbf{x}}$ since the transmitted and the reflected wave are travelling in opposite directions, $v_1 = c/n_1$ and $v_2 = c/n_2$ are the phase velocities of the waves in media 1 and media 2 and finally

$$\begin{cases} \hat{\mathbf{n}}_T = \cos\theta_T \hat{\mathbf{y}} + \sin\theta_T \hat{\mathbf{z}} \quad , \\ \hat{\mathbf{n}}_R = \cos\theta_R \hat{\mathbf{y}} + \sin\theta_R \hat{\mathbf{z}} \quad , \end{cases} \tag{3}$$

are the polarization vectors of the transmitted and the reflected wave.
The boundary conditions at the surface of separation of the two media for the parallel components of the electric fields and the parallel components of the magnetic fields, are the following

$$\begin{cases} E_I \hat{\mathbf{n}}_I + E_R \hat{\mathbf{n}}_R = E_T \hat{\mathbf{n}}_T \quad , \\ \dfrac{E_I}{\mu_1 v_1}(\hat{\mathbf{k}}_I \times \hat{\mathbf{n}}_I) + \dfrac{E_R}{\mu_1 v_1}(\hat{\mathbf{k}}_R \times \hat{\mathbf{n}}_R) = \dfrac{E_T}{\mu_2 v_2}(\hat{\mathbf{k}}_T \times \hat{\mathbf{n}}_T) \quad , \end{cases} \tag{4}$$

where $\hat{\mathbf{k}}_I = \hat{\mathbf{x}}$, $\hat{\mathbf{n}}_I = \hat{\mathbf{y}}$,

$$\begin{cases} \hat{\mathbf{k}}_R \times \hat{\mathbf{n}}_R = -\cos\theta_R \hat{\mathbf{z}} + \sin\theta_R \hat{\mathbf{y}} \quad , \\ \hat{\mathbf{k}}_T \times \hat{\mathbf{n}}_T = \cos\theta_T \hat{\mathbf{z}} - \sin\theta_T \hat{\mathbf{y}} \quad . \end{cases} \tag{5}$$

and $\mu v = \sqrt{\mu/\varepsilon} = Z$ is the impedence of the media.

Projecting the boundary conditions (4) along the $\hat{\mathbf{z}}$ we get

$$\begin{pmatrix} E_R & -E_T \\ Z_2 E_R & Z_1 E_T \end{pmatrix} \begin{pmatrix} \sin\theta_R \\ \sin\theta_T \end{pmatrix} = 0 \quad . \tag{6}$$

Since the determinant of the 2×2 matrix is different from zero then $\sin\theta_R = \sin\theta_T = 0$. This shows that the polarizations of the reflected and transmitted wave (3) must be parallel to the polarization of the incident wave ($\hat{\mathbf{y}}$).

Choosing $\theta_R = \theta_T = 0$ we get

$$\begin{cases} E_I + E_R \cos\theta_R = E_T \cos\theta_T \quad , \\ Z_2(E_I - E_R \cos\theta_R) = Z_1 E_T \cos\theta_T \quad , \end{cases} \tag{7}$$

from which follows

$$\begin{cases} E_R = \left(\dfrac{1-\beta}{1+\beta}\right) E_I \quad , \\ E_R = \left(\dfrac{2}{1+\beta}\right) E_I \quad , \end{cases} \tag{8}$$

where $\beta = Z_1/Z_2$. So choosing $\theta_R = \theta_T = 0$, if $\beta < 1$ and $E_I > 0$ then $E_R, E_T > 0$ (convention usually used).

$\boxed{\text{-Problem 3-}}$ A wave in air is normally incident on a coated piece of glass (see figure 2 below). The air has an index of refraction $n_1 = 1$, the coating

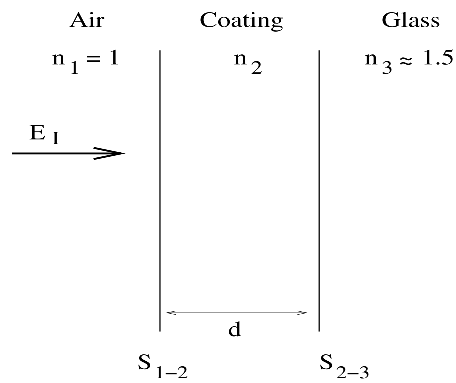

Figure 2: Non reflecting film.

has n_2, and the glass has n_3. The coating has a thickness, d, and the air and glass are infinitely thick. The wavelength of the wave in air is λ. What condition must n_2 and d satisfy in order to minimize the reflected wave (i.e., in terms of n_1, λ, and n_3)?
HINT: the two reflected waves must exactly cancel.

SOLUTION

We want to construct a *non reflecting film*. We can increase the fraction of light reflected by a glass surface (correspondingly increasing the fraction of transmitted light) by evaporating on glass a thin film of a transparent material (the so called "coated lenses" are created this way).

Since we are assuming that there isn' t any metallic deposit on the surface of the two boundaries, no phase change different from 0 or π occour at reflection (in the reflected wave) or refraction (in the transmitted wave) at the two separation surfaces S_{1-2} or S_{2-3}. We shall denote by $\tau_{1,2}$ the ratio of the transmitted to the incident amplitude when the wave passes from the air to the coating and by $\tau_{2,3}$ the ratio of the transmitted to the incident amplitude when the wave passes from the coating to the glass. Let $\rho_{1,2}$ and $r23$ be the corresponding ratios of the reflected to the incident amplitudes and let A be the amplitude of the incident wave ($\mathbf{E}_I$ in figure 2). For the case of normal incidence on the boundary of two linear materials i and j with

$\mu_i = \mu_j = \mu_o$ (the case of the problem) we have

$$\tau_{i,j} = \frac{2n_i}{n_i + n_j} \quad , \qquad \rho_{i,j} = \frac{n_i - n_j}{n_i + n_j} \quad . \tag{1}$$

The incident wave $\mathbf{E}_I$ traveling in the air will be partly transmitted in the wave $\mathbf{E}_T$ traveling in the coating. $\mathbf{E}_T$ will be partly transmitted in the wave $\mathbf{E}_{T_1}$ traveling in the glass and partly reflected through an internal reflection into a wave remaining into the coating. The successive internal reflections of the wave trapped into the coating will generate a series of waves $E_{T_2}, \ldots, E_{T_n}, \ldots$ transmitted into the glass (see the schematic representation of figure 3). Thus the various transmitted waves at S_{2-3} are represented by

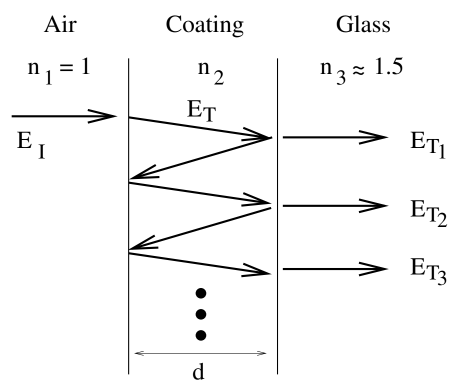

Figure 3: Schematic representation of the internal reflections.

$$\begin{aligned}
E_{T_1} &= \tau_{1,2}\tau_{2,3}Ae^{i\omega t} \quad , & (2)\\
E_{T_2} &= \tau_{1,2}\tau_{2,3}(\rho_{2,1}\rho_{2,3})Ae^{i\omega t-\alpha} \quad , & (3)\\
&\vdots \\
E_{T_n} &= \tau_{1,2}\tau_{2,3}(\rho_{2,3}\rho_{2,1})^{(n-1)}Ae^{i[\omega t-(n-1)\alpha]} = \tau_{1,2}\tau_{2,3}Ae^{i\omega t}(\rho_{2,3}\rho_{2,1}e^{-i\alpha})^{n-1} & (4)\\
&\vdots
\end{aligned}$$

where we have indicated with α the phase difference between neighboring waves, namely

$$\alpha = 2dk_c = 2d\left(\frac{2\pi}{\lambda_c}\right) = 2d\left(\frac{n_2 2\pi}{\lambda}\right) \tag{5}$$

where k_c is the modulus of the wave vector and λ_c the wavelength of the wave propagating in the coating [1]. The resultant transmitted wave is

$$E_{T_{tot}} = E_{T_1} + E_{T_1} + \ldots + E_{T_1} + \ldots = \tau_{1,2}\tau_{2,3}Ae^{i\omega t}\sum_{n=1}^{\infty}(\rho_{2,3}\rho_{2,1}e^{-i\alpha})^{n-1} \quad . \quad (7)$$

Since $|\rho_{2,3}\rho_{2,1}e^{-i\alpha}| < 1$ then

$$E_{T_{tot}} = \frac{\tau_{1,2}\tau_{2,3}}{1-\rho_{2,3}\rho_{2,1}e^{-i\alpha}}Ae^{i\omega t} \quad . \quad (8)$$

Calling I_I the intensity of the incident wave on air (E_I) I_T the intensity of the transmitted wave on glass ($E_{T_{tot}}$) and I_R the intensity of the transmitted wave on glass ($E_{R_{tot}}$, which can be computed by a similar procedure as $E_{T_{tot}}$), we have for the conservation of energy

$$I_I = I_R + I_T \quad . \quad (9)$$

Then minimizing I_R equals to maximizing I_T. From the result (8) follows that

$$\frac{I_T}{I_I} = n_3\left(\frac{(\tau_{1,2}\tau_{2,3})^2}{1+(\rho_{2,3}\rho_{2,1})^2 - 2(\rho_{2,3}\rho_{2,1})\cos\alpha}\right) \quad . \quad (10)$$

To maximize I_T we have to minimize the denominator of the left hand side of equation (10). We have two cases.

-i- When $1 < n_2 < n_3$ then $\rho_{2,3}\rho_{2,1} < 0$ then the maximum of I_T appear when $\cos\alpha = -1$, i.e. when $\alpha = (2n+1)\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. From equation (5) follows that in this case the condition that n_2 and d must satisfy in order to minimize the reflected wave is

$$2d = \frac{\lambda}{n_2}(n + \frac{1}{2}) \quad (11)$$

[1] To determine how the wavelength of the wave change going from media i to media j is sufficient to remember that at the interface the frequency of the wave traveling with velocity $v_i = c/n_i$ and wave vector k_i on media i has to equal the frequency of the wave traveling with velocity $v_j = c/n_j$ and wave vector k_j on media j, namely

$$k_i v_i = k_j v_j \quad \Rightarrow \quad \frac{\lambda_j}{\lambda_i} = \frac{v_j}{v_i} = \frac{n_i}{n_j} \quad . \quad (6)$$

-ii- When $1 < n_3 < n_2$ then $\rho_{2,3}\rho_{2,1} > 0$ then the maximum of I_T appear when $\cos\alpha = 1$, i.e. when $\alpha = 2n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. From equation (5) follows that in this case the condition that n_2 and d must satisfy in order to minimize the reflected wave is

$$2d = \frac{\lambda}{n_2} n \tag{12}$$

In this cases I_R will reduce to

$$\frac{I_T}{I_I} = n_3 \frac{(\tau_{1,2}\tau_{2,3})^2}{(1 - \rho_{2,3}\rho_{2,1})^2} \sim (\tau_{1,2}\tau_{2,3})^2 \quad , \tag{13}$$

where we have used the fact that usually $\rho_{2,3}\rho_{2,1} \ll 1$. This approximation justify the HINT of the problem.

One can show that the index of refraction n_2 of the thin film must be intermediate between those of the air and of the glass in order to have I_T minimized for a wide range of wavelength λ centered around the value given in equation (11).

It' s important to observe that in our case $(1 \neq n_3)$ I_R will never vanish. That is possible only when $n_1 = n_3$ [2].

[2]This case is analyzed for example in "Optics" by Bruno Rossi par.3-10 page 135.