Homework # 3 Solutions

set of units used: MKSA

-Problem 1- Consider this static vector potential given in spherical coordinates

$$
\begin{cases}
A_r = 0 \\
A_\theta = 0 \\
A_\phi = \dfrac{g}{r} \tan\left(\dfrac{\theta}{2}\right)
\end{cases}
\tag{1}
$$

Note that $\mathbf{A}$ is not defined at $\theta = \pi$ (the $-\hat{z}$ axis). Calculate the $\mathbf{B}$ field that this $\mathbf{A}$ describes.

SOLUTION

We have to take the curl in spherical coordinates of the vector potential

$$
\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\ A_\phi) - \frac{\partial}{\partial\phi}A_\theta\right]\hat{r} +
\tag{2}
$$

$$
\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} +
\tag{3}
$$

$$
\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right]\hat{\phi}\ \ .
\tag{4}
$$

Inserting the vector potential (1) one gets

$$
\mathbf{B} = \nabla \times \mathbf{A} = \frac{g}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\tan(\theta/2)\right)\hat{r} = \frac{g}{r^2}\hat{r}\ \ .
\tag{5}
$$

One immediately recognizes in $\mathbf{B}$ the magnetic field of a magnetic monopole.

COMMENT: **the meaning of the singularity**. The vector potential given in the problem can be obtained from the following argument

$$
\mathbf{A}(\mathbf{r}) = \int_L d\mathbf{A}(\mathbf{r})\ \ ,
\tag{6}
$$

where

$$d\mathbf{A}(\mathbf{r}) = -g\mathbf{dl}' \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \quad , \tag{7}$$

is the vector potential in $\mathbf{r}$ due to an elementary magnetic dipole $\mathbf{dm} = g\mathbf{dl}'$ in $\mathbf{r}'$, and L is the negative $\hat{z}$ axis.

   This construction suggests the following picture for a monopole: the magnetic charge g is the ending point of a "string" of elementary magnetic dipoles which extends to infinity as shown in figure 1. The vector potential gener-
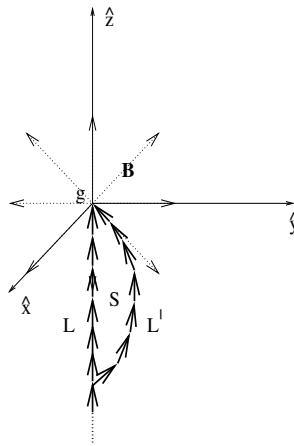

Figure 1: Description of a magnetic monopole g. S is a surface contained between the two strings L and L'.

ated by the magnetic charges g is singular on the string. as we know. This definition [1] allows to use for the electromagnetic interactions necessary to describe the dynamics and statics of a monopole, the same structure of the ones used in the decription of the electric monopole, $\mathbf{B} = \nabla \times \mathbf{A}$, etc. ... In particular the strange fact that the total flux through a closed surface surrounding a magnetic charge g is $4\pi g$ (which wouldn' t allow to use $\nabla \mathbf{B} = 0$) is prevented assuming the existence of a magnetic field $\mathbf{B}'$ very intense on the string of dipoles and zero outside. $\mathbf{B}'$ is there just to compensate the flux exiting from the edge of of the string: the monopole. The magnetic field due

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[1] Used by Dirac in his original argument to show how the electric charge quantization is a necessary consequenc of the assumption of the axistence of the magnetic monopole.

to the magnetic charge will be then

$$\mathbf{B}_{monopole} = \nabla \times \mathbf{A} - \mathbf{B}' \quad . \tag{8}$$

One can show that the freedom in choosing the string of dipoles from a point at infinite and the monopole is equivalent to the freedom due to the gauge tranformations. In particular changing the string L to the string L' (see figure 1) produces the following change in the vector potential describing the monopole

$$\mathbf{A}_{L'}(\mathbf{r}, t) = \mathbf{A}_L(\mathbf{r}, t) + g\nabla\Omega_S(\mathbf{r}, t) \tag{9}$$

where $\Omega_S(\mathbf{r}, t)$ is the solid angle with vertex in $\mathbf{r}$ which sees the surface S (see figure ??). Try to derive formula (9) using equation (6) as an exercise !

-Problem 2- Suppose that a current **j** flows radially outward from a very small region at the origin

$$\mathbf{j} = \frac{J}{4\pi} \frac{\hat{r}}{r^2} \tag{1}$$

(a) Calculate the time derivative of the charge density, ρ, at some radius $r \neq 0$.

(b) Calculate the time derivative of the total charge contained in the small region at the origin.

(c) Currents produce magnetic fields, don' t they ? Use Ampere' s law to calculate the magnetic field at some $r \neq 0$. Which direction does it point ? (Hint: this is a trick question.)

SOLUTION

Let' s start with a **mathematical note**: We know from electrostatic that the more general solution of the equation [1]

$$\nabla^2 \phi(\mathbf{r}) = -4\pi\delta(\mathbf{r}) \quad , \tag{2}$$

is, apart for a constant, the Coulomb potential

$$\phi(\mathbf{r}) = \frac{1}{|\mathbf{r}|} + constant \quad , \tag{3}$$

thus,

$$\mathbf{E} = -\nabla\phi(\mathbf{r}) = \frac{\hat{r}}{|\mathbf{r}|^2} \quad . \tag{4}$$

On the other way around we can say that (4) is the more general solution of the equation

$$\nabla\mathbf{E} = 4\pi\delta(\mathbf{r}) \quad , \tag{5}$$

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[1]$\delta(\mathbf{r})$ is the Dirac delta function which physically represents the charge number density of a charge distribution composed by one charge on the origin. It is mathematically defined by the following properties:

i. $\delta(\mathbf{r}) = 0$ for $\mathbf{r} \neq 0$

ii. $\int_{\Omega} \delta(\mathbf{r})d\mathbf{r} = 1$ for every neighbourhoood Ω of the origin.

when we look for a solution which can be written as the gradient of some function.

Coming to the problem, for the current density in eq. (1) one has

(a) From the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \mathbf{j} = -J\delta(\mathbf{r}) \quad , \tag{6}$$

then $\partial \rho / \partial t = 0$ for $r \neq 0$.

(b) Integrating equation (6) over the small region at the origin containing the total charge Q one gets

$$\frac{\partial Q}{\partial t} = -J \quad . \tag{7}$$

(c) The Ampere Maxwell' s law tells

$$\nabla \times \mathbf{B} = \mu_o(\mathbf{j} + \mathbf{j}_D) \quad , \tag{8}$$

where we have indicated with $\mathbf{j}_D = \varepsilon_o[\partial E/\partial t]$ the displacement current. Taking the gradient of this eq. we get

$$\nabla \mathbf{j}_D = -\nabla \mathbf{j} = -J\delta(\mathbf{r}) \quad . \tag{9}$$

Since we know that exist ϕ such that $\mathbf{j}_D = \nabla \phi$ (in particular $\phi = \varepsilon_o[\partial E/\partial t]$ the mathematical note tells us that

$$\mathbf{j}_D = -J \quad , \tag{10}$$

from which follows $\nabla \times \mathbf{B} = 0$ and consequently $\mathbf{B} = 0$.

-Problem 3- A very long solenoid of radius a, with N turns per unit lenght, carries a current I_s. Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire, with resistance R. When the current in the solenoid is gradually decreased, a current I_r is induced in the ring.

(a) Calculate I_r, in terms of dI_s/dt.

(b) The power $(I_r^2 R)$ delivered to the ring must have come from the solenoid. Confirm this by calculating the Poynting vector just outside the solenoid (the *electric field* is due to the changing flux in the solenoid; the *magnetic field* is due to the current in the ring). Integrate over the entire surface of the solenoid, and check that you recover the correct total power.
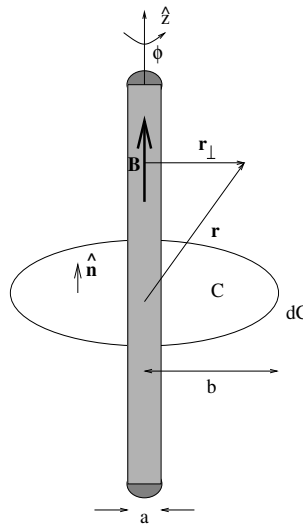
SOLUTION

Figure 2: The apparatus described in the problem.

(a) The magnetic field inside the solenoid is

$$\mathbf{B}(t) = \mu_o N I_s(t)\hat{z} \quad . \tag{1}$$

Since $B(t)$ is varying with time an electric field with will be present with circles concentric with the solenoid as field lines. Then using Faraday's law one can determine the electric potential felt by the electrons in the circular ring of wire ∂C (see figure 2)

$$V = \oint_C \mathbf{E} \cdot \mathbf{dl} = 2\pi b E = -\frac{d}{dt} \int_{\partial C} \mathbf{B} \cdot \mathbf{da} = -\pi a^2 \mu_o N \frac{dI_s}{dt} \quad . \qquad (2)$$

Ohm law gives then for the current in the circular ring of resistance R

$$I_r = \frac{\mu_o N \pi a^2}{R} \frac{dI_s}{dt} \quad . \qquad (3)$$

and for the power dissipated through Joule effect

$$P_r = R I_r^2 = I_r \mu_o N \pi a^2 \frac{dI_s}{dt} \quad . \qquad (4)$$

(b) Let's now calculate the Poynting vector just outside the solenoid. The electric field is due to the changing magnetic field inside the solenoid. Near the solenoids it will be

$$\mathbf{E} = -\mu_o \frac{\mu_o N \pi a}{2} \frac{dI_s}{dt} \hat{\phi} \quad . \qquad (5)$$

The magnetic field is due to the current induced in the ring. Near the solenoids surface $r_\perp \approx a \ll b$, with $r_\perp$ the component of $\mathbf{r}$ orthogonal to $\hat{z}$, we have (see eq.(3) in problem 3 of Homework #1 Solutions)

$$\mathbf{B} \approx -\frac{\mu_o}{2} \frac{b^2}{(z^2 + b^2)^{3/2}} I_r \hat{z} \quad . \qquad (6)$$

Using equations (5) and (6) one can construct the Poynting vector (see figure 3)

$$\mathbf{S} = \frac{1}{\mu_o}(\mathbf{E} \times \mathbf{B}) = I_r \frac{\mu_o N a b^2}{4} \frac{1}{(z^2 + b^2)^{3/2}} \hat{r_\perp} \quad . \qquad (7)$$

The power P_r of eq. (4) dissipated in the resistor must come from the solenoid. Then integrating the Poynting vector over the surface of the solenoids we expect to get again P_r.

$$P = \int \mathbf{S} \cdot \mathbf{da} = I_r \frac{\mu_o N \pi a^2 b^2}{2} \int_{-\infty}^{\infty} \frac{1}{(z^2 + b^2)^{3/2}} \, dz = P_r \quad . \qquad (8)$$
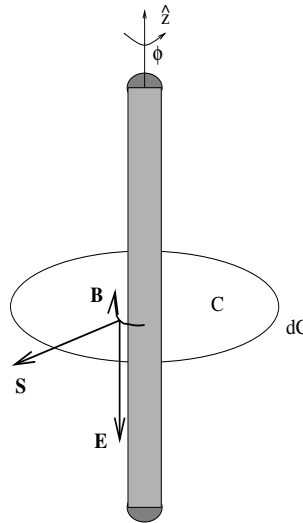
Figure 3: The Poynting vector near the surface of the solenoid.

-Problem 4- Show that it is always possible to choose

$$\nabla \mathbf{A} = -\mu_o \varepsilon_o \frac{\partial V}{\partial t} \quad , \tag{1}$$

as required for the Lorentz gauge, assuming you know how to solve the non homogeneous wave eqation. Is it always possible to pick $V = 0$? How about $A = 0$?

SOLUTION

It is alwaus possible, through a gauge transformation, choose the potentials $\mathbf{A}$ and V in such a way that they satisfy the Lorentz gauge (1). Suppose you have the potentials that don' t satisfy the Lorentz gauge. Through a gauge transformation

$$\begin{cases} \mathbf{A}(\mathbf{r}, t) \to \mathbf{A}'(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \nabla \chi(\mathbf{r}, t) \\ V(\mathbf{r}, t) \to V'(\mathbf{r}, t) = V(\mathbf{r}, t) - \dfrac{\partial}{\partial t} \chi(\mathbf{r}, t) \end{cases} \tag{2}$$

we get the potentials in the new gauge $\mathbf{A}'$ and V' which describe the same physics as $\mathbf{A}$ and V; i.e. the electric and magnetic fields don' t change

(are invariant under the gauge transformation). If we now impose that the potentials in the new gauge satisfy the Lorentz gauge (1) we get

$$\nabla \mathbf{A}' + \frac{1}{c^2}\frac{\partial V'}{\partial t} = 0 = \nabla \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} + \left(\nabla^2 \chi - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2}\right) \quad . \tag{3}$$

This shows that if we are able to find a gauge function $\chi(\mathbf{r}, t)$ which satisfies the equation

$$\nabla^2 \chi - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} = -(\nabla \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}) \quad , \tag{4}$$

then the new potentials will be in the Lorentz gauge.

It is not always possible to have a gauge in which $V' = 0$. Immagine that you have found a gauge χ in which that condition holds. This means that in the preceding gauge one had $V = \partial\chi/\partial t$ then taking the gradient, $\mathbf{E} = -\nabla(\partial\chi/\partial t)$ and finally taking the curl, $\partial\mathbf{B}/\partial t$. This means that is possible to find a gauge in which $V' = 0$ only if originally we were in a situation in which the magnetic field wasn' t varying with time.

It is not always possible to have a gauge in which $\mathbf{A}' = 0$. Immagine that you have found a gauge χ in which that condition holds. This means that in the preceding gauge one had $\mathbf{A} = -\nabla\chi$ then $\mathbf{B} = 0$. This means that is possible to find a gauge in which $A' = 0$ only if originally were in a situation in which the magnetic field was zero.