Homework # 11 Solutions

set of units used: MKSA

-Problem 1- A particle of charge q moves in a circle of radius R at a constant angular velocity ω. (Assume that the circle lies in the xy plane, centered at the origin, and at time $t = 0$ the charge is at $(R, 0)$, on the positive x axis.) Find the Liénard-Wiechert potentials for points on the z axis.

SOLUTION

The Liénard Wiechert potentials for a moving point charge q can be written as follows

$$V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_o} \frac{q}{|\mathbf{r} - \mathbf{r}'| - (\mathbf{r} - \mathbf{r}') \cdot \mathbf{v}/c} \quad , \tag{1}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t) \quad , \tag{2}$$

where we have:

$$\begin{aligned}
\mathbf{r} &\equiv \text{point of observation} = (0, 0, z) \quad , \\
\mathbf{r}' &\equiv \text{retarded position of the charge} \\
&= (R\cos\omega t_r, R\sin\omega t_r, 0) \quad , \\
|\mathbf{r} - \mathbf{r}'| &= \sqrt{z^2 + R^2} \quad , \\
\mathbf{v} &\equiv \text{retarded velocity of the charge} \\
&= (-R\omega\sin\omega t_r, R\omega\cos\omega t_r, 0) \quad , \\
(\mathbf{r} - \mathbf{r}') \cdot \mathbf{v} &= 0 \quad , \\
t_r &\equiv \text{retarded time} = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} = t - \frac{\sqrt{z^2 + R^2}}{c} \quad .
\end{aligned}$$

In the above eqs. R is the radius of the circle traveled by the charge at a constant angular velocity ω. Collecting all the previous results we get for the scalar potential

$$V(z, t) = \frac{q}{4\pi\varepsilon_o} \frac{1}{\sqrt{z^2 + R^2}} \quad , \tag{3}$$

which is the same result as in the static case. For the vector potential we get

$$\mathbf{A}(z,t) = \frac{R\omega}{4\pi\varepsilon_o c^2}\frac{q}{\sqrt{R^2+z^2}}\{-\hat{\mathbf{x}}\sin[\omega(t-\sqrt{z^2+R^2}/c)] \\ +\hat{\mathbf{y}}\cos[\omega(t-\sqrt{z^2+R^2}/c)]\} \quad . \tag{4}$$

-Problem 2- Suppose an electron decelerates at a constant rate a from some initial velocity v_o down to zero.

(a) What fraction of its initial kinetic energy is lost to radiation? (The rest is absorbed by whatever mechanism keeps the acceleration constant.) Assume $v_o \ll c$ so thet Larmor formula can be used.

(b) To get a sense of the numbers involved, suppose the initial velocity is thermal (around $10^5 m/s$) and the distance the electron goes is $30\mathring{A}$. What can you conclude about radiation losses for the electrons in an ordinary conductor?

SOLUTION

(a) The electron has an initial energy $E_o = (1/2)mv_o^2$. It decelerate at a constant rate a down to zero velocity and since $v_o \ll c$ we can approximately describe its radiation power through the Larmor formula

$$\frac{dE}{dt} = -\frac{1}{4\pi\varepsilon_o}\frac{2}{3}\frac{q^2}{c^3}a^2 \quad . \tag{1}$$

The above eq. can be easily integrated since its right hand side is a constant in time. Calling ΔE the energy lost to radiation during the deceleration of the charge we get

$$\Delta E = \frac{1}{4\pi\varepsilon_o}\frac{2}{3}\frac{q^2}{c^3}a^2\Delta t \quad , \tag{2}$$

where Δt is the time interval in which the partice velocity changes from v_o to 0. Since the motion of the charge is uniformly decelerated $\Delta t = v_o/a$ and we can rewrite eq. (2) as

$$\Delta E = \frac{1}{4\pi\varepsilon_o}\frac{2}{3}\frac{q^2}{c^3}av_o \quad , \tag{3}$$

and the fraction of initial kinetic energy lost to radiation will be

$$\frac{\Delta E}{E_o} = \frac{1}{3\pi\varepsilon_o}\frac{q^2}{mc^3}\frac{a}{v_o} \quad . \tag{4}$$

(b) If the electron has an initial velocity $v_o = 10^5 m/s$ and travels a distance $d = 30\mathring{A} = 30 \times 10^{-10} m$ before coming to rest $\Delta t = 2d/v_o$ and $a = v_o/\Delta t = v_o^2/2d$. We can then rewrite eq. (4) in terms of the known quantities as follows

$$\frac{\Delta E}{E_o} = \frac{1}{6\pi\varepsilon_o}\frac{q^2}{mc^3}\frac{v_o}{d} = 2.1 \times 10^{-10} \quad . \tag{5}$$

Consider for example a piece of iron at room temperature. The number density of free electron in it is

$$n \sim 17 \times 10^{22} cm^{-3} \quad , \tag{6}$$

the resistivity (at $373K$) is

$$\rho \sim 14.7 \times 10^{-6} \Omega cm \quad , \tag{7}$$

the average velocity of the electrons (the fermi velocity)

$$v \sim 1.98 cm/s \quad . \tag{8}$$

Using Ohm' s law we can then write the fraction of initial kinetic energy lost to heat the iron as

$$\frac{\Delta E}{E_o} = \frac{[\rho(nev)^2]d/v}{n[(1/2)mv^2]} = \frac{2\rho ne^2 d}{mv} \sim 3.5 \quad . \tag{9}$$

A comparison of this result with eq. (4) shows that in ordinary conductors the energy loss due to radiation is negligible with respect to the one due to the resistivity.

-Problem 3- In Bohr' s theory of hydrogen, the electron in its ground state was supposed to travel in a circle of radius $5 \times 10^{-11} m = 0.5 \AA$, held in orbit by the Coulomb attraction of the proton. According to classical electrodynamics, the electron should radiate, and hence spiral to the nucleus. Show that $v \ll c$ for most of the trip (so you can use the Larmor formula), and calculate the lifespan of Bohr' s atom. (Assume each revolution is essentially circular.)

SOLUTION

We want to calculate the *lifetime* of the electron in the Bohr' s description of the hydrogen atom if we apply to it the laws of the classical elettrodynamics. We will make the following three hypotheses (the consistency of these hypotheses with the following derivation of the lifespan of the electron will be tested only for the third one as requested by the problem.)

(i.) The initial conditions are such that the electron would describe a circular orbit if it wouldn' t radiate (not elliptical, ...)

(ii.) The electron radiate and spiral into the nucleus. When calculating the acceleration of the electron we will approximate, at each point, the spiral as the tangent circle passing through the given point. This approximation is a good approximation when the energy lost by the electron in one turn around the nucleus is much smaller then the energy of the electron itself. (this is sometimes called the adiabatic hypotheses.)

(iii.) We will assume that the speed of the electron $v \ll c$ while traveling along most of the spiral. This allow us to use Larmor formula (exact only in the rest frame of the particle) to describe the electron energy loss due to radiation.

The power radiated by the accelerating electron is given by Larmor formula (hypotheses (iii.))

$$\frac{dE}{dt} = -\frac{1}{4\pi\varepsilon_o}\frac{2}{3}\frac{q^2}{c^3}|\mathbf{a}|^2 \quad , \tag{1}$$

where

$$E = \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_o}\frac{e^2}{r} \quad . \tag{2}$$

Equating (see hypotheses (i.) and (ii.)) the Coulomb attraction and the centripetal force we obtain the dependence of the energy from r, namely

$$m\frac{v^2}{r} = \frac{1}{4\pi\varepsilon_o}\frac{e^2}{r^2} \quad , \tag{3}$$

which gives [1]

$$\frac{1}{2}mv^2 = \frac{1}{2}\left[\frac{1}{4\pi\varepsilon_o}\frac{e^2}{r}\right] \quad \Rightarrow \tag{4}$$

$$E = -\frac{1}{2}\left[\frac{1}{4\pi\varepsilon_o}\frac{e^2}{r}\right] \quad , \tag{5}$$

$$v = \sqrt{\frac{e^2}{(4\pi\varepsilon_o)mr}} \quad . \tag{6}$$

Inserting $m = 9.11 \times 10^{-31}kg$, $e = 1.60 \times 10^{-19}coul$, and $\varepsilon_o = 8.85 \times 10^{-12}coul^2/Nm^2$ into eq. (6) we get that the electron will approach the speed of light only at a radius $r \sim 2.82 \times 10^{-15}m = 2.82fm$ [2]. This shows that hypotheses (iii.) is reasonable: for most of the trip along the spiral the alectron velocity is much smaller then the speed of light.

From eq. (3) we get the acceleration of the electron as

$$a = \frac{1}{4\pi\varepsilon_o}\frac{e^2}{mr^2} \quad . \tag{7}$$

Taking the derivative of eq. (5) with respect to time and substituting the result and eq. (7) into Larmor formula (eq. (1)) we get

$$\frac{1}{4\pi\varepsilon_o}\frac{e^2}{2r^2}\frac{dr}{dt} = -\left(\frac{1}{4\pi\varepsilon_o}\right)^2 \frac{2}{3}\frac{q^2}{c^3}\frac{e^4}{m^2r^4} \quad . \tag{8}$$

Integrating this eq. between R and the nucleus radius r_o we get the lifespan τ of the electron as

$$\tau = \int_0^\tau dt = -\frac{3(4\pi\varepsilon_o)^2}{4}\frac{m^2c^3}{e^4}\int_R^{r_o} r^2 dr \quad . \tag{9}$$

Since $R \gg r_o$ ($R = 0.5\mathring{A}$ and $r_o \sim 1fm$) we get

$$\tau = \frac{(4\pi\varepsilon_o)^2}{4}\frac{m^2c^3R^3}{e^4} \simeq 1.32 \times 10^{-11}s \quad . \tag{10}$$

[1] Note that this is nothing else than the virial theorem for Coulomb force: $T = -V/2$ $\Rightarrow E = T + V = V/2$.

[2] The nucleus radius is of the order of magnitude of $1fm$.

-Problem 4- A nonrelativistic electron with initial speed v_o is aimed directly at a repulsive Coulomb field $V(r) = -Ze/r$ (i.e., a negative point charge $Q = -Ze$). The electron decelerates, comes to rest, and accelerates outward as it returns to infinity. Show that the final kinetic energy of the electron is approximately:

$$\frac{1}{2}mv_f^2 = \frac{1}{2}mv_o^2\left(1 - \frac{16v_o^3}{45Zc^3}\right) \quad .$$

(This is H&M, problem 8-12).

SOLUTION

For a nonrelativistic electron $v \ll c$ so we can *approximately* describe its loss of energy by radiation, through the Larmor formula (which holds exactly only in the rest frame of the particle in the framework of classical electrodynamics), namely

$$\frac{dE}{dt} = -\frac{1}{4\pi\varepsilon_o}\frac{2}{3}\frac{q^2}{c^3}\dot{v}^2 = -\alpha\,\dot{v}^2 \quad , \tag{1}$$

where E is the total energy of the particle. It will be usefull to rewrite eq. (1) as

$$\frac{dE}{dv} = \frac{dE}{dt}\frac{dt}{dv} = -\alpha\,\dot{v} \quad . \tag{2}$$

The problem can be treated as a one dimensional problem where the total energy of the charged particle is given by its kinetic energy and its interaction with the coulomb field generated by a charge $-ze$ at the origin, namely

$$E = \frac{1}{2}mv^2 - \frac{ze^2}{x} \quad . \tag{3}$$

The particle comes from $x = \infty$ with an initial total energy

$$E_o = \frac{1}{2}mv_o^2 \quad , \tag{4}$$

decelerates, come at rest after loosing energy ΔE by radiation and return to infinity loosing again an energy ΔE by radiation and with an kinetic energy

$$E_f = \frac{1}{2}mv_f^2 \quad , \tag{5}$$

[1] Since at $x = \infty$ the potential energy of the charged particle due to the Coulomb field of the scattering center is zero we have

$$2\Delta E = E_f - E_o = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2 \quad . \tag{6}$$

In order to solve the problem we have to find ΔE and to show that

$$2\Delta E = E_o \left[\frac{16v_o^3}{45Zc^3} \right] \tag{7}$$

From eq. (2) and using Newton' s law we get

$$\frac{dE}{dv} = -\alpha a = -\alpha \frac{ze^2}{mx^2} \quad . \tag{8}$$

Next we assume a *small* radiation by the nonrelativistic particle and approximate $E \sim E_o$ in eq. (3), namely solving for x

$$x = \frac{2ze^2}{mv^2 - 2E} \quad . \tag{9}$$

Sobstituting eq. (9) into eq. (8) we obtain a differential equation which integrated gives

$$\begin{aligned} \Delta E &= \int_{E_o}^{E_f} dE = -\frac{\alpha}{4ze^2} \int_{v_o}^{0} dv(m^2v^4 - 4mv^2E + 4E^2) \\ &= \frac{\alpha}{4ze^2} \left(m^2 \frac{v_o^5}{5} - mE_o \frac{4v_o^3}{3} + 4E_o^2 v_o \right) \\ &= \frac{\alpha}{4ze^2} \left(\frac{2}{5} - \frac{4}{3} + 2 \right) E_o v_o^3 = \frac{\alpha}{4ze^2} \frac{16}{15} E_o v_o^3 \quad , \end{aligned} \tag{10}$$

and recolling the definition of α in eq. (1) we get in the end

$$\Delta E = E_o \frac{1}{2} \left[\frac{16v_o^3}{45Zc^3} \right] \quad , \tag{11}$$

which is the desired result (compare with eq. (7)).

[1]From the Larmor formula follows that the power radiated by the particle is proportional to the square of its acceleration, and the deceleration in approaching the scattering center must be equal in absolute value to the acceleration in leaving it since in both travels the force acting on the particle is the same.