

Homework # 1 Solutions

all problems are from D.J.Griffiths: "introduction to electrodynamics" unless stated
set of units used: MKSA

-SOLUTION to problem 1.13-

Given the following definitions

$$\begin{aligned}\mathbf{r} &\equiv (x - x_o)\hat{\mathbf{x}} + (y - y_o)\hat{\mathbf{y}} + (z - z_o)\hat{\mathbf{z}} \quad , \\ r &\equiv |\mathbf{r}| \equiv \sqrt{(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2} \quad , \\ \hat{\mathbf{r}} &\equiv \frac{\mathbf{r}}{r} \quad , \\ \nabla &\equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \quad .\end{aligned}$$

it follows:

(a)

$$\begin{aligned}\nabla r^2 &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \right) [(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2] \\ &= \hat{\mathbf{x}}\frac{\partial}{\partial x} [(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2] \\ &\quad + \hat{\mathbf{y}}\frac{\partial}{\partial y} [(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2] \\ &\quad + \hat{\mathbf{z}}\frac{\partial}{\partial z} [(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2] \\ &= \hat{\mathbf{x}}2(x - x_o) + \hat{\mathbf{y}}2(y - y_o) + \hat{\mathbf{z}}2(z - z_o) \\ &= 2\mathbf{r} \quad ,\end{aligned}$$

(b)

$$\begin{aligned}\nabla \frac{1}{r} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \right) \frac{1}{\sqrt{(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2}} \\ &= \hat{\mathbf{x}} \left(-\frac{x - x_o}{[(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2]^{3/2}} \right)\end{aligned}$$

$$\begin{aligned}
& + \hat{\mathbf{y}} \left(-\frac{y - y_o}{[(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2]^{3/2}} \right) \\
& + \hat{\mathbf{z}} \left(-\frac{z - z_o}{[(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2]^{3/2}} \right) \\
= & -\frac{\mathbf{r}}{r^3} = -\frac{\hat{\mathbf{r}}}{r^2} \quad ,
\end{aligned}$$

(c)

$$\begin{aligned}
\nabla r^n &= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) [(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2]^{n/2} \\
&= \hat{\mathbf{x}} n(x - x_o) [(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2]^{(n-2)/2} \\
&\quad + \hat{\mathbf{y}} n(y - y_o) [(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2]^{(n-2)/2} \\
&\quad + \hat{\mathbf{z}} n(z - z_o) [(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2]^{(n-2)/2} \\
&= \mathbf{r} n r^{n-2} = \hat{\mathbf{r}} n r^{n-1} \quad .
\end{aligned}$$

It's worthwhile to observe at this point that the following simple rule generally hold for the gradient of a function of r

$$\nabla f(r) = \hat{\mathbf{r}} \frac{df}{dr}(r) \quad .$$

-SOLUTION to problems 1.15 (a),(b); 1.19 (a),(b)-

(a) Given $\mathbf{v} = (x^2, 3xz^2, -2xz)$ one gets for the divergence

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz) \\ &= 2x - 2x = 0 \quad ,\end{aligned}$$

and for the curl

$$\begin{aligned}\nabla \times \mathbf{v} &\equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{\mathbf{x}} \left[\frac{\partial}{\partial y}(-2xz) - \frac{\partial}{\partial z}(3xz^2) \right] \\ &\quad + \hat{\mathbf{y}} \left[\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(-2xz) \right] \\ &\quad + \hat{\mathbf{z}} \left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial y}(x^2) \right] \\ &= \hat{\mathbf{x}}(-6xz) + \hat{\mathbf{y}}(2z) + \hat{\mathbf{z}}(3z^2) \quad ,\end{aligned}$$

(b) Given $\mathbf{v} = (xy, 2yz, 3zx)$ one gets for the divergence

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx) \\ &= y + 2z + 3x \quad ,\end{aligned}$$

and for the curl

$$\begin{aligned}\nabla \times \mathbf{v} &= \hat{\mathbf{x}} \left[\frac{\partial}{\partial y}(3zx) - \frac{\partial}{\partial z}(2yz) \right] \\ &\quad + \hat{\mathbf{y}} \left[\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(3zx) \right] \\ &\quad + \hat{\mathbf{z}} \left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy) \right] \\ &= \hat{\mathbf{x}}(-2y) + \hat{\mathbf{y}}(-3z) + \hat{\mathbf{z}}(-x) \quad .\end{aligned}$$

-SOLUTION to problem 1.31-

We have to test the divergence theorem for the function $\mathbf{v} = (xy, 2yz, 3zx)$.

$$\int_V \nabla \cdot \mathbf{v} \, d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a} \quad , \quad (1)$$

where V is the volume of the cube shown in fig. 1 and S is its surface. We

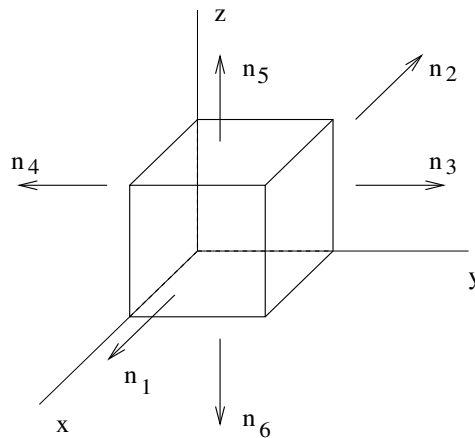


Figure 1: cube of side 2. \mathbf{n}_i is the unitary vector orthogonal to the i^{th} face.

will calculate separately the right hand side (RHS) and the left hand side (LHS) of equation (1) and show that they are equal:

(LHS) As calculated in the solution to problem 1.15 (b) $\nabla \cdot \mathbf{v} = y + 2z + 3x$.

$$\begin{aligned} \int_V \nabla \cdot \mathbf{v} \, d\tau &= \int_V (y + 2z + 3x) \, dx dy dz \\ &= \int_0^2 dz \int_0^2 dy \int_0^2 dx (y + 2z + 3x) \\ &= \int_0^2 dz \int_0^2 dy (2y + 4z + 6) \\ &= \int_0^2 dz (4 + 8z + 12) = 8 + 16 + 24 = 48 \end{aligned}$$

(RHS)

$$\oint_S \mathbf{v} \cdot d\mathbf{a} = \sum_i \int_{i^{th} \text{ face}} \mathbf{v} \cdot \mathbf{n}_i \, da$$

$$\begin{aligned} &= \left(\int_{3^{\text{rd}} \text{ face}} 2yz \, dx dz - \int_{4^{\text{th}} \text{ face}} 2yz \, dx dz \right) \\ &\quad + \left(\int_{5^{\text{th}} \text{ face}} 3zx \, dx dy - \int_{6^{\text{th}} \text{ face}} 3zx \, dx dy \right) \\ &\quad + \left(\int_{1^{\text{st}} \text{ face}} xy \, dy dz - \int_{2^{\text{nd}} \text{ face}} xy \, dy dz \right) \\ &= \int_0^2 \int_0^2 2y \, dy dz + \int_0^2 \int_0^2 4z \, dx dz + \int_0^2 \int_0^2 6x \, dx dy \\ &= \int_0^2 4 \, dz + \int_0^2 8 \, dx + \int_0^2 12 \, dy \\ &= 8 + 16 + 24 = 48 \end{aligned}$$

-SOLUTION to problem 1.33-

We have to test Stokes' theorem for the function $\mathbf{v} = (xy, 2yz, 3zx)$.

$$\int_S \nabla \times \mathbf{v} \, d\mathbf{a} = \oint_L \mathbf{v} \cdot d\mathbf{l} \quad , \quad (1)$$

where S is the triangular shaded area shown in fig. 2 and L is its boundary.

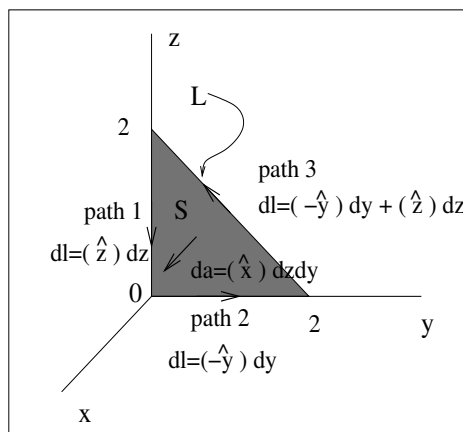


Figure 2: Triangular surface S .

We will calculate separately the right hand side (RHS) and the left hand side (LHS) of equation (1) and show that they are equal:

(LHS) As calculated in the solution to problem 1.15 (b) $\nabla \times \mathbf{v} = (-2y, -3z, -x)$.

$$\begin{aligned} \int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} &= \int_S (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{x}} dz dy = \int_S (-2y) \hat{\mathbf{x}} dz dy \\ &= \int_0^2 dy \int_0^{2-y} dz (-2y) = \int_0^2 2y(y-2) dy \\ &= \int_0^2 (4y - 2y^2) dy = \frac{2}{3} 8 - 8 = -\frac{8}{3} \quad , \end{aligned}$$

(RHS)

$$\oint_L \mathbf{v} \cdot d\mathbf{l} = \sum_{i=1}^3 \int_{\text{path-}i} \mathbf{v} \cdot d\mathbf{l} \quad (2)$$

$$= \left[\int_2^0 \mathbf{v}|_{y=0} \hat{\mathbf{z}} dz + \int_0^2 \mathbf{v}|_{z=0} (-\hat{\mathbf{y}}) dy \right] \quad (3)$$

$$+ \int_{\text{path-3}} \mathbf{v}|_{y+z=2} (-\hat{\mathbf{y}} dy + \hat{\mathbf{z}} dz) \Big]_{x=0} \quad (4)$$

$$= \left[0 + 0 + \int_2^0 2y(2-y) dy \right] \quad (5)$$

$$= \int_0^2 (4y - 2y^2) dy = \frac{2}{3}8 - 8 = -\frac{8}{3} . \quad (6)$$

-SOLUTION to problem 2.1 (a), (b)-

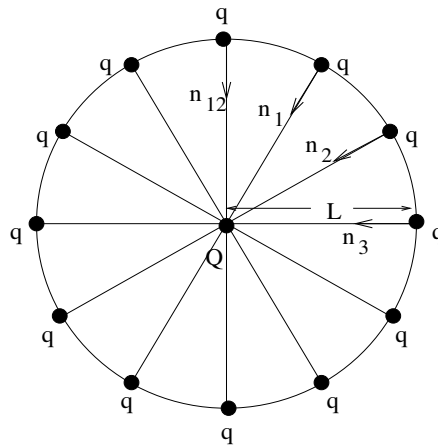


Figure 3: shows twelve equal charges, q , fixed at the corners of a regular 12-sided polygon (one on each numeral of a clock face). \mathbf{n}_i is the versor pointing the center of the polygon from the i^{th} charge. A test charge Q is at the center of the polygon.

- (a) Given the arrangement of charges depicted in fig. 3 and assuming fixed the charges q at the corners of the 12-sided regular polygon the force exerted by the i^{th} -charge (i.e. the charge at “i o’clock”) on a test charge Q at the center of the polygon can be written as

$$\mathbf{F}_i = \frac{1}{4\pi\epsilon_0} \frac{Qq}{L} \mathbf{n}_i \quad ,$$

where L is half diagonal of the polygon and \mathbf{n}_i is the unit vector pointing the test charge from the i^{th} -charge. The total force exerted on Q is then

$$\mathbf{F} = \sum_{i=1}^{12} \mathbf{F}_i = \frac{1}{4\pi\epsilon_0} \frac{qQ}{L} \sum_{i=1}^{12} \mathbf{n}_i \quad (1)$$

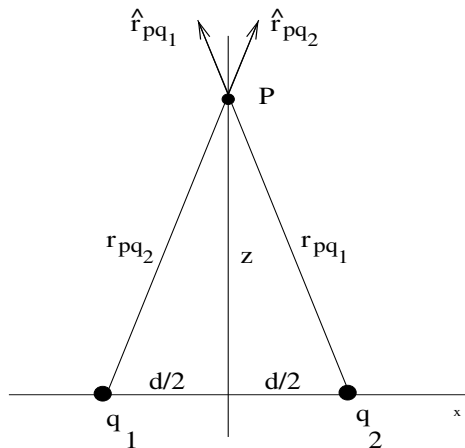
$$= \frac{qQ}{L} \sum_{i=1}^6 (\mathbf{n}_i + \mathbf{n}_{i+6}) = \frac{qQ}{L} \sum_{i=1}^6 (\mathbf{n}_i - \mathbf{n}_i) = 0 \quad . \quad (2)$$

(b) when we remove the charge at 6 o'clock the force on Q will be

$$\mathbf{F} = \sum_{\substack{i=1 \\ i \neq 6}}^{12} \mathbf{F}_i = -\mathbf{F}_6 = \frac{1}{4\pi\epsilon_0} \frac{Qq}{L} \mathbf{n}_6 \quad ,$$

as follows from eq. (1).

-SOLUTION to problem 2.2-

Figure 4: two charges q_1 and q_2 of equal charge q .(a) When $q_1 = q_2 = q$

$$\begin{aligned}
 \mathbf{E}_p &= \frac{1}{4\pi\epsilon_o} \left(\frac{q}{r_{pq1}^2} \hat{\mathbf{r}}_{pq1} + \frac{q}{r_{pq2}^2} \hat{\mathbf{r}}_{pq2} \right) = \frac{1}{4\pi\epsilon_o} \frac{q}{z^2 + d^2/4} (\hat{\mathbf{r}}_{pq1} + \hat{\mathbf{r}}_{pq2}) \\
 &= \frac{1}{4\pi\epsilon_o} \frac{q}{z^2 + d^2/4} \left(\hat{\mathbf{z}} \frac{2z}{\sqrt{z^2 + d^2/4}} \right) = \frac{1}{4\pi\epsilon_o} \frac{2zq}{(z^2 + d^2/4)^{3/2}} \hat{\mathbf{z}} \\
 &= \frac{1}{4\pi\epsilon_o} \frac{2q}{z^2(1 + (d/z)^2/4)^{3/2}} \hat{\mathbf{z}} \xrightarrow{d/z \ll 1} \hat{\mathbf{z}} \frac{1}{4\pi\epsilon_o} \left(\frac{2q}{z^2} + O((d/z)^2) \right) .
 \end{aligned}$$

In the limit $z \gg d$ the dominant term in \mathbf{E}_p resembles the electric field generated in P by a charge $2q$ at the origin.

(b) when $q_1 = -q_2 = q$

$$\begin{aligned}
 \mathbf{E}_p &= \frac{1}{4\pi\epsilon_o} \left(\frac{q}{r_{pq1}^2} \hat{\mathbf{r}}_{pq1} - \frac{q}{r_{pq2}^2} \hat{\mathbf{r}}_{pq2} \right) = \frac{1}{4\pi\epsilon_o} \frac{q}{z^2 + d^2/4} (\hat{\mathbf{r}}_{pq1} - \hat{\mathbf{r}}_{pq2}) \\
 &= \frac{1}{4\pi\epsilon_o} \frac{q}{z^2 + d^2/4} \left(\hat{\mathbf{x}} \frac{d}{\sqrt{z^2 + d^2/4}} \right) = \frac{1}{4\pi\epsilon_o} \frac{qd}{(z^2 + d^2/4)^{3/2}} \hat{\mathbf{x}}
 \end{aligned}$$

$$= \frac{1}{4\pi\epsilon_o} \frac{qd}{z^3(1 + (d/z)^2/4)^{3/2}} \hat{\mathbf{x}} \xrightarrow{d/z \ll 1} \hat{\mathbf{x}} \frac{1}{4\pi\epsilon_o} \left(\frac{qd}{z^3} + O((d/z)^2) \right) .$$

In the limit $z \gg d$ the dominant term in \mathbf{E}_p resembles the electric field generated in P by a “pure” dipole $-qd\hat{\mathbf{x}}$ at the origin ¹.

¹The assembly of charge used in this part (b) of the problem is called a “physical” dipole of magnitude $\mathbf{p} = -qd\hat{\mathbf{x}}$. The “pure” dipole is *defined* as a physical dipole with q infinitely big ($q \rightarrow e(\infty)$) and d infinitely small in such a way to keep the product $qd = p$ a finite quantity. p is the magnitude of the dipole.

-SOLUTION to problem 2.5-

The electric field at a point P on the axis of a circular loop of radius r carrying a uniform line charge λ (see fig. 5) can be calculated as follows.

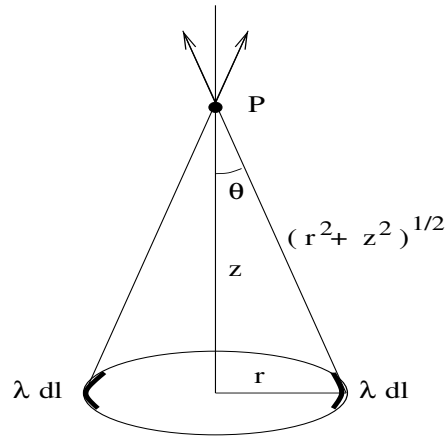


Figure 5: loop of radius r carrying a uniform line charge λ .

First consider the electric field due to two diametral opposite elementary pieces dl of the circular loop

$$d\mathbf{E}_p = \frac{1}{4\pi\epsilon_0} \frac{2 \cos \theta}{r^2 + z^2} (\lambda dl) \hat{\mathbf{z}} \quad ,$$

$$\cos \theta = \frac{z}{\sqrt{r^2 + z^2}} \quad ,$$

then sum over half-loop to obtain the electric field due to all the uniformly charged loop

$$\begin{aligned} \mathbf{E}_p &= \int_{1/2\text{-loop}} d\mathbf{E}_p = \hat{\mathbf{z}} \int_{1/2\text{-loop}} \lambda dl \frac{1}{4\pi\epsilon_0} \frac{2z}{(r^2 + z^2)^{3/2}} \\ &= \hat{\mathbf{z}} \frac{1}{4\pi\epsilon_0} \frac{2z\lambda}{(r^2 + z^2)^{3/2}} \int_{1/2\text{-loop}} dl = \hat{\mathbf{z}} \frac{1}{4\pi\epsilon_0} \frac{2\pi r \lambda z}{(r^2 + z^2)^{3/2}} \quad . \end{aligned}$$

-SOLUTION to problem 2.10-

Using Gauss's law we know that the flux of the electric field due to the

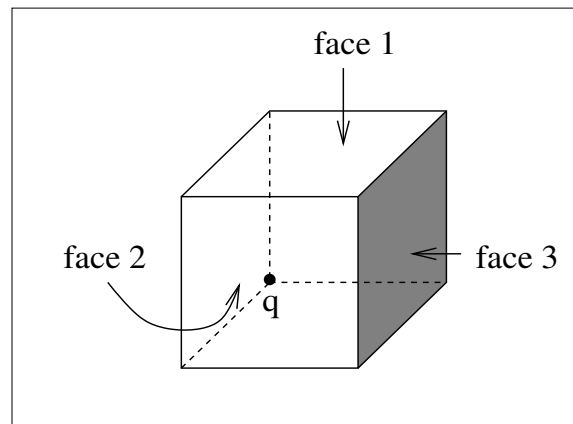


Figure 6: cube with a charge q at one of its corners.

charge q placed at one of the corner of the cube of fig. 6 through a surface enclosing the charge must be equal to q/ϵ_o . The flux of the electric field, for how it is defined ($\int \mathbf{E} \cdot d\mathbf{a}$), is different from zero only on the faces 1, 2 and 3 of the cube (see fig. 6; on the other 3 faces the electric field component orthogonal to the face vanishes) and for the symmetry of the problem has an *equal value* on each of the three faces. Since only $1/8$ of the total flux q/ϵ_o goes through the cube one can then conclude that the flux of the electric field through the shaded face must be

$$\int_{\text{face-3}} \mathbf{E} \cdot d\mathbf{a} = \frac{q}{(8 * 3)\epsilon_o} .$$