Fractional Statistics in Two Dimensions

The notion of statistics is usually related to the sign that a many-body wavefunction aquires when two particles are interchanged. If the wavefunction gets a plus sign, we say that it describes a bosonic system; if it gets a minus sign, we say that it describes a fermionic system. Actually we can give a more general definition of statistics: let $\psi(1,2)$ be the wavefunction describing two identical *hard core* particles with definite angular momentum, and let us assume that when we move particle 2 around particle 1 by an azimuthal angle $\Delta\phi$ (see fig. 1), the wavefunction changes according to [1],

$$\psi(1,2) \rightarrow \psi'(1,2) = e^{i\nu\Delta\phi}\psi(1,2) \quad . \tag{1}$$

The phase acquired by the wavefunction depends on a parameter ν which is usually called *statistics*. The meaning of ν and of (1) becomes more clear if we consider the exchange of the two particles. This can be realized in two ways:

i) Moving particle 2 around particle 1 by an angle $\Delta\phi = \pi$, and then performing a rigid translation of the center of mass to reach the initial spatial configuration (see fig. 2a);

ii) Moving particle 2 around particle 1 by an angle $\Delta\phi = -\pi$, and then performing a rigid translation of the center of mass to reach the initial spatial configuration (see fig. 2b).

In the first case the wavefunction aquires a phase $\exp(i\pi\nu)$, whilst in the second case, it gets a phase $\exp(-i\pi\nu)$. This simple example shows that there is a dramatic difference between two and three or more dimensions as far as statistic is concerned. In fact in $d \geq 3$ there is no intrinsic difference between case *i*) and case *ii*) since we can always deform the transformation in *i*) into the one in *ii*) in a continuous way: for example we can lift the path

[1] $\psi(1,2)$ is an eigenvector of $J_z^{2,1}/\hbar$ the z-component of the angular momentum operator of particle 2 relative to particle 1, with eigenvalue ν

in $i)$ into the third dimension and fold it back onto the plane superposing it to the path in $ii)$ [2].

The consequences of this observation are very important: if $i)$ and $ii)$ are the same physical operation, the wavefunction must change in the same way under the two operations, which means that in $d \geq 3$ one must have,

$$e^{i\pi\nu} = e^{-i\pi\nu} \quad . \tag{2}$$

Clearly (2) can only be true if $\nu = 0, 1 \bmod 2$. So in $d \geq 3$ the statistics cannot be arbitrary. Under the exchange of two particles, the wavefunction picks up either a plus sign if $\nu = 0$ (bosonic statistics) or a minus sign if $\nu = 1$ (fermionic statistic). There are no other possibilities.

The situation changes drastically in two dimensions where it is not any more possible to deform continuously the path in $i)$ into the one in $ii)$ since by assumption the particles cannot go through each other. Hence in $d = 2$, $i)$ and $ii)$ are two topologically and physically distinct operations. The equality (2) doesn' t necessarily hold any more and the statistical parameter ν can be arbitrary, at least in principle [3]. Particles with this property are called *anyons*. From this example we learn another important fact: in $d = 2$ it is not enough to specify the initial and final configurations to completely characterize the system; it is also necessary to specify how the different tra-

[2]For example in $d = 3$ if the third dimension is sufficiently extended we can always make a rotation of π around the axis connecting the two particles with which we superpose the path in $i)$ to the path in $ii)$.

[3]There are restrictions on ν coming from the topology of the two dimensional space. For example for particles moving on a torus (or a 2D box with periodic boundary conditions), ν can only be a rational number.

jectories wind or *braid* around each other. In other words the time evolution of the particles is important and cannot be neglected in $d = 2$. As we will see later this fact implies that in order to classify and characterize anyons, the representations of the permutation group must be replaced by those of the more complicated *braid group*.

One of the essential features of anyons is the violation of the discrete symmetries of parity P and time reversal T. This is particularly evident in our two particles example. In fact, under a parity or time reversal transformation, the phase acquired by the wavefunction $\psi(1,2)$ changes according to [4],

$$e^{\pm i\nu\pi} \to e^{\mp i\nu\pi} \quad , \tag{3}$$

for cases $i)$ and $ii)$ respectively, and for $\nu \neq 0, 1$ a violation of P and T occurs. This breaking of the discrete symmetries is the hallmark of anyonic statistics and is the signal that experiments look for to demonstrate the presence (or absence) of anyons in two dimensional phenomena.

Anyonic Statistics Problem

Let M_N^d be the configuration space of a collection of N identical hard core particles in d dimensions and let q and q' be two arbitrary points in M_N^d. According to the standard path integral formulation of quantum mechanics the amplitude for the system to evolve from the configuration q at time t to the configuration q' at time t' is given by the kernel,

$$\rho(q', t'; q, t) = \iint_{q(t)=q}^{q(t')=q'} e^{\frac{i}{\hbar}\int_t^{t'} d\tau \mathcal{L}[q(\tau),\dot{q}(\tau)]} \mathcal{D}q \quad , \tag{4}$$

where $\mathcal{L}(q, \dot{q})$ is the lagrangian density for the N particle system and the symbol $\iint_{q(t)=q}^{q(t')=q'} \mathcal{D}q$ denotes the sum over all paths connecting q at time t to q' at time t'. The kernel $\rho(q', t'; q, t)$ evolves the single valued wavefunction $\psi(q, t)$ according to,

$$\begin{aligned} \psi(q', t') &= \int_{M_N^d} dq \langle q', t' | q, t \rangle \langle q, t | \psi \rangle \\ &= \int_{M_N^d} dq \rho(q', t'; q, t) \psi(q, t) \quad . \end{aligned} \tag{5}$$

[4] Under P or T, $J_z^{1,2} \to -J_z^{1,2}$. Then if $R = \exp[i(J_z^{1,2}/\hbar)\pi]$ we have for example $RP\psi = PP^\dagger RP\psi = P\exp[-i(J_z^{1,2}/\hbar)\pi]\psi = \exp[-i\nu\pi]P\psi$.

Without loss of generality, we choose $q = q'$ and hence loops in M_N^d. Two lops are considered equivalent (or homotopic) if one can be obtained from the other by a continuous deformation. All homotopic loops are grouped into one class and the set of all such classes is called the *fundamental group* and is denoted by π_1 [5]. Thus an element of $\pi_1(M_N^d)$ is simply the set of all loops in M_N^d which can be continuously deformed into each other. On the other hand, loops belonging to two different elements of $\pi_1(M_N^d)$ cannot be connected by a continuous transformation. With this in mind we can organize the sum over all loops in (4) into a sum over homotopic classes $\alpha \in \pi_1(M_N^d)$ and into a path integral in each class. Therefore (4) may be rewritten as,

$$\begin{aligned} \rho(q,t';q,t) &= \sum_{\alpha\in\pi_1(M_N^d)} \chi(\alpha)\rho_\alpha(q,t';q,t) \qquad (6) \\ &= \sum_{\alpha\in\pi_1(M_N^d)} \chi(\alpha) \iint_{q_\alpha(t)=q}^{q_\alpha(t')=q} e^{\frac{i}{\hbar}\int_t^{t'} d\tau \mathcal{L}[q_\alpha(\tau),\dot{q}_\alpha(\tau)]} \mathcal{D}q_\alpha \quad . \end{aligned}$$

In order for (6) to make sense as a probability amplitude, the complex weights $\chi(\alpha)$ cannot be arbitrary. In fact, since we want to mantain the usual rule for combining probabilities,

$$\rho(q',t';q,t) = \int_{M_N^d} dq_o \rho(q',t';q_o,t_o)\rho(q_o,t_o;q,t) \quad , \qquad (7)$$

the weights $\chi(\alpha)$ must satisfy,

$$\chi(\alpha_1)\chi(\alpha_2) = \chi(\alpha_1 \cdot \alpha_2) \quad , \qquad (8)$$

for any α_1 and α_2. Equation (8) can also be read as the statement that $\chi(\alpha)$ must be a one dimensional unitary ($|\chi|^2 = 1$) representation of the fundamental group $\pi_1(M_N^d)$ [6]. To see which representations are possible, we have to specify better what are M_N^d and its fundamental group.

M_N^d and its fundamental group

[5] In the set π_1 one can define a product $\cdot$ in a very simple and natural way: if α_1 and α_2 are two classes with representatives loops $q_1(t)$ and $q_2(t)$, then $\alpha_1 \cdot \alpha_2$ is the class whose representative is the loop $q_1 q_2(t)$ (that is the loop q_1 followed by the loop q_2). It can be shown that this product furnishes π_1 with a group structure.

[6] M. G. G. Laidlaw and M. De Witt, Phys. Rev. D **3**, 1375 (1971)

Consider a systen of N identical hard core particles mooving in the euclidean d-dimensional space, $I\!R^d$. A configuration of such a system is clearly specified by the N coordinates of the particles, that is by an element of $(I\!R^d)^N$. However because of the hard core assumption, any two particles cannot occupy the same position. So from $(I\!R^d)^N$ we have to remove the diagonal,

$$\Delta = \{(\mathbf{r}_1, \ldots, \mathbf{r}_N) \in (I\!R^d)^N : \mathbf{r}_i = \mathbf{r}_j \text{ for some } i \neq j\} \quad . \tag{9}$$

Furthermore our particles are identical and indistinguishable, so we should identify configurations which differ only in the ordering of the particles. In other words we should divide by the permutation group S_N. Therefore we conclude that the configuration space for our system is

$$M_N^d = \frac{(I\!R^d)^N - \Delta}{S_N} \quad . \tag{10}$$

To find the fundamental group of such space is a standard problem in algebraic topology, which was solved in the early 60' s [7]. It turns out that the fundamental group of M_N^d is given by [8],

$$\pi_1(M_N^d) = \begin{cases} S_N & \text{if } d \geq 3 \\ B_N & \text{if } d = 2 \end{cases} \tag{11}$$

where B_N is Artin' s braid group of N objects which contains the permutation group S_N as a finite subgroup [9].

Even from this formal point of view we see that there is a crucial difference between two and three or more dimensions, exactly as we discovered in the simple example at the beginning of the discussion. To have a more explicit understanding of (11), let us consider our two particle example in the light of what we have just observed. Let us start with the case of two dimensions. Instead of assigning the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ for the two particles, is more convenient to introduce the center of mass coordinate,

$$\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2) \quad \in I\!R^2 \quad , \tag{12}$$

and the relative coordinate,

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \quad \in I\!R^2 - 0 \quad . \tag{13}$$

[7] E. Fadell and L. Neuwirth, Math. Scand. **10**, 111 (1962)
[8] In $d = 1$ the configuration space M_N^1 is not connected.
[9] E. Artin, Abh. Math. Sem. Hamburg **4**, 47 (1926);Annals of Math. **48**, 101 (1947)

(We have removed the origin because of the hard core requirement.) Since $\mathbf{R}$ is invariant under the permutations of S_2, we can write,

$$M_2^2 = I\!R^2 \times r_2^2 \quad , \tag{14}$$

where r_2^2 is some space describing the two degrees of freedom of the relative motion. We now argue that r_2^2 has the topology of a cone. Since two configurations which differ only in the ordering of the particle indeces are indistinguishable, $\mathbf{r}$ and $-\mathbf{r}$ must be identified. The space r_2^2 is then the upper half plane without the origin and with the positive x-axis identified with the negative one, i.e. is a cone without the tip (see fig. 3).

According to the decomposition (14), any loop in M_2^2 can be classified by the number of times it winds around the cone r_2^2. Two loops q and q' with different winding numbers are homotopically inequivalent: it is not possible to deform one into the other since the vertex of the cone has been removed. Thus the space r_2^2 and $I\!R^2 \times r_2^2$, are infinitely connected, and,

$$\pi_1(M_2^2) = \pi_1(I\!R^2 \times r_2^2) = \mathbb{Z} = \mathsf{B}_2 \quad . \tag{15}$$

It is important to realize that if the vertex of the cone were included (i.e. allowing particles to occupy the same position in space) the configuration space would be simply connected. Any loop, even when winding around the cone, would be contracted to a point by deforming and unwinding it through the tip. Thus, if we do not impose the hard core constraint on the particles, we can describe only bosonic statistics.

Let us now turn to the case of two particles in three dimensions. After introducing the center of mass coordinate $\mathbf{R} \in I\!R^3$, we can decompose the configurations space as,

$$M_2^3 = I\!R^3 \times r_2^3 \quad , \tag{16}$$

where the space r_2^3 describes the three degrees of freedom of the relative motion. These are the length and the two angles of the relative coordinate

$\mathbf{r}$. As before $\mathbf{r}$ and $-\mathbf{r}$ are identified. It is easy to realize that r_2^3 is just the product of the semi-infinite line describing $|\mathbf{r}|$ and the projective space $\mathcal{P}_2$ describing the orientation of $\pm\mathbf{r}/|\mathbf{r}|$. In turn $\mathcal{P}_2$ can be described as the northern hemisphere with opposite points on the equator being identified (see fig. 4).

The space $\mathcal{P}_2$ is doubly connected and admits two classes of loops: those which can be shrunk to a point by a continuous transformation and those which cannot. In fig. 5 we exibit a typical contractible loop q_1 and a typical non-contractible loop q_2.

Therefore from the decomposition (16) and the topology of r_2^3, we deduce that,

$$\pi_1(M_2^3) = \pi_1(\mathbb{R}^3 \times r_2^3) = \mathbb{Z}_2 = \mathsf{S}_2 \quad . \tag{17}$$

Thus only bosons and fermions can exist, the former corresponding to contractible loops and the latter to non-contractible loops.

We have seen that at the heart of the anyonic statistics there is the braid group B_N in place of the permutation group S_N which is responsible for ordinary statistics. There are only two one dimensional unitary representations of S_N, namely the identical one, χ(even and odd permutations) $= +1$ (bosonic statistics) and the alternating one, χ(even permutations) $= +1$, χ(odd permutations) $= -1$ (fermionic statistics). Whereas the braid group admits a whole variety of one dimensional [10] unitary representations whose

[10] When dealing with non-scalar quantum mechanics, i.e. when the wavefunctions are multiplets instead of one component objects as assumed in the discussion, appropriate higher dimensional representations of $\pi_1(M_N^d)$ would be necessary.

labelling parameter will be identified with the statistics ν.

The Braid Group

The braid group of N strands, B_N, is an infinite group which is generated by $N-1$ elementary moves $\sigma_1, \ldots, \sigma_{N-1}$ satisfying,

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad , \tag{18}$$

for $i = 1, \ldots, N-2$, and

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad , \tag{19}$$

for $|i-j| \geq 2$. The inverse of σ_i is denoted by σ^{-1}, the identity $\mathbb{1}$, and the center of B_N is generated by $(\sigma_1 \cdots \sigma_{N-1})^N$. To describe the elementary moves σ_i, it is convenient to introduce a pictorial representation as follows. Given N vertical strands, the generator σ_i acts on them by braiding the $i-th$ strand around the $(i+1)-th$ in a definite way, as shown in fig. 6. The inverse generator σ^{-1} is represented by the move shown in fig. 7. For example for $N = 3$ the braid represented in fig. 8 corresponds to $\sigma_1^3 \sigma_2 \sigma_1$ (generators on the right act first).

We remark that in general $\sigma_i^2 \neq \mathbb{1}$. If $\sigma_i^2 = \mathbb{1}$ for all i, then the braid group reduces to the permutation group S_N, which is a finite group.

Recalling that a closed path in M_N^2 can be represented by N world-lines in the three space {x,y,t} with no intersections and with the final position in $\mathbb{R}^2$ at t' being just permutations of the initial ones at t, diagrams like those in figs. 6, 7, 8 can also be interpreted as the projection on the x-t plane say, of those world-lines. This is explicitly shown in the following example with three particles. Suppose that at time t their configuration is the one displayed in fig. 9 We can describe it by listing the azimuthal angle of all

possible pairs of particles measured with respect to some arbitrary reference axes. In the specific case of fig. 9 we have,

$$\begin{aligned} \varphi_{12}(t) &= 0 \quad , \\ \varphi_{13}(t) &= \eta \quad , \\ \varphi_{23}(t) &= \xi \quad . \end{aligned} \tag{20}$$

The symbol φ_{ij} denotes the azimuthal angle of particle j with respect to particle i, namely

$$\varphi_{ij} = \tan^{-1}\left(\frac{x_j^2 - x_i^2}{x_j^1 - x_i^1}\right) \quad , \tag{21}$$

$(x_i^1 - x_i^2)$ being the cartesian coordinates of the $i-th$ particle.

Let us then suppose that at time t' the particles reach the position shown in fig. 10. Now the winding angles are,

$$\varphi_{12}(t') = \xi + \pi \quad , \tag{22}$$

$$\varphi_{13}(t') = \eta + \pi \quad , \tag{23}$$

$$\varphi_{23}(t') = \pi \quad .$$

The configuration at time t and the one at time t' are the same even if particle 1 and 3 have been interchanged. This is because the particles are identical. Notice that the following is always true,

$$\sum_{i<j}[\varphi_{ij}(t') - \varphi_{ij}(t)] = n\pi \quad , \tag{24}$$

where n is an integer (3 in our case). This can be interpreted by saying that to complete a loop in configuration space an integer number of exchanges is always necessary. We can see explicitly this fact in our example if we place fig. 10 on top of fig. 9 and draw lines to connect the initial and final positions of each particle. In a side view this is represented by the braiding of fig. 11, which is $\sigma_1\sigma_2\sigma_1$ in terms of the elementary moves. We see that indeed

there are $n = 3$ generators, i.e. 3 exchanges. Notice that if we considered the braiding $\sigma_1^3\sigma_2\sigma_1$ (of fig. 8) the final configuration would look the same as in fig. 11, but the winding angles would be different (specifically $\varphi_{12}(t')$ would be $\xi + 3\pi$). If we just look at the particle positions on the plane, there is no way of distinguishing between these two cases. They correspond to the same permutation of the initial position, but if $\sigma_i^2 \neq 1\!\!1$, they are dynamically different. The actual briding of the particles, and not simply their permutations, are the important things to be considered in discussing anyonic statistics.

Let us now come back to the problem of finding the one dimensional unitary representations of the bridal group. These are given by,

$$\chi(\sigma_k) = e^{-i\nu\pi} \quad , \tag{25}$$

for any $k = 1, \ldots, N-1$. ν is the statistics, a parameter defined modulo 2 already introduced at the beginning of the discussion. Since in general $\sigma_k^2 \neq 1\!\!1$, ν is an arbitrary number. In the elementary move σ_k all angles φ_{ij} remain constant except for $\varphi_{k,k+1}$ which changes by π. Thus we can rewrite $\chi(\sigma_k)$ as,

$$\chi(\sigma_k) = e^{-i\nu\Delta\varphi_{k,k+1}} = e^{-i\nu\sum_{i<j}\Delta\varphi_{ij}^{(k)}} \quad , \tag{26}$$

where we have introduced,

$$\Delta\varphi_{ij}^{(k)} \equiv \varphi_{ij}^{(k)}(t') - \varphi_{ij}^{(k)}(t) \equiv \pi\delta_{i,k}\delta_{j,k+1} \quad , \tag{27}$$

to represent the change of the winding angles in the move σ_k. Using this notation, (26) can be easily generalized to an arbitrary braiding α, and hence we can write,

$$\chi(\alpha) = exp\left[-i\nu\sum_{i,j}\int_t^{t'} d\tau \frac{d}{d\tau}\varphi_{ij}^{(\alpha)}(\tau)\right] \quad , \tag{28}$$

where the increment $\Delta\varphi_{ij}^{(\alpha)}$ has been written as an integral over the imaginary time. Notice that the functions $\varphi_{ij}^{(\alpha)}(\tau)$ are in general very complicated and can be specified only when the dynamics of the particles is fully taken into account. However the formal definition (28) may come usefull when inserted into the density matrix expression (6),

$$\rho(q,t';q,t) = \sum_{\alpha\in\pi_1(M_N^2)} \iint_{q_\alpha(t)=q}^{q_\alpha(t')=q} e^{\left\{\frac{i}{\hbar}\int_t^{t'} d\tau \left[\mathcal{L}[q_\alpha(\tau),\dot{q}_\alpha(\tau)] - \hbar\nu\sum_{i,j}\frac{d\varphi_{ij}^{(\alpha)}(\tau)}{d\tau}\right]\right\}} \mathcal{D}q_\alpha \ .$$

This expression tells us that instead of dealing with anyons governed with the Lagrangian $\mathcal{H}$, we can work with bosons whose dynamics is dictated by the new Lagrangian $\mathcal{L}' = \mathcal{L} - \hbar\nu\sum_{i,j} d\varphi_{ij}^{(\alpha)}(\tau)/d\tau$. Thus one can trade anyons with some kind of "ficticious" force and describe anyons as ordinary particles (bosons for example) with an additional statistical interaction. Notice that this statistical interaction is intrinsically topological in nature, in our example it is actually a total derivative. Its addition to the original Hamiltonian $\mathcal{L}$ doesn' t change the equations of motion, which are a reflection of the *local* structure of the configurational space, but does change the statistical properties of the particles, which are instead related to the *global* topological structure of the configuration space.

Statistical Mechanics

All statistic properties of a quantum system described by an Hamiltonian $\hat{\mathcal{H}}$, and in thermal equilibrium at the inverse temperature β are obtainable from the thermal density matrix operator,

$$\hat{\rho} = \exp(-\beta\hat{\mathcal{H}}) \quad . \tag{29}$$

In the configurations space representation the thermal density matrix can be written using path integral notation,

$$\rho(q',q;\beta) = \sum_{\alpha\in\pi_1(M_N^d)} \chi(\alpha) \iint_{q_\alpha(0)=q}^{q_\alpha(\hbar\beta)=q'} e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau\mathcal{H}(q_\alpha(\tau),\dot{q_\alpha}(\tau))}\mathcal{D}q_\alpha \quad , \tag{30}$$

where $\mathcal{H}(q, \dot{q})$ is the classical Hamiltonian of the N hard core, identical particles.

One is usually interested in calculating the partition function of the system which is given by the trace of the density matrix. So, like in the quantum mechanic case, we choose $q = q'$, or loops in M_N^d. The discussion proceeds exactly like for the quantum mechanics case except for the fact that now $\rho(q, q; \beta)$ has to be a real positive probability function. This means that we have to look for one dimensional unitary representations $\chi(\alpha)$ of the fundamental group.

$$\chi(\alpha) = e^{-i\nu n_\alpha \pi} \quad , n_\alpha \text{ integer } . \tag{31}$$

In $d \geq 3$ there are only 2 possible representations of the permutation group: the one corresponding to the bosonic statistics ($\nu = 0$ mod 2) and the one corresponding to the fermionic statistics ($\nu = 1$ mod 2). In $d = 2$ one has to choose representations of the braid group. Those are labelled by the statistics parameter ν and can be formally expressed as

$$\chi(\alpha) = exp\left[-i\nu \sum_{i,j} \int_0^{\hbar\beta} d\tau \frac{d}{d\tau} \varphi_{ij}^{(\alpha)}(\tau)\right] \quad , \tag{32}$$

so that the expression for the diagonal of the density matrix gets the suggestive form,

$$\begin{aligned} \rho(q, q; \beta) &= \sum_{\alpha \in \pi_1(M_N^2)} \chi(\alpha) \rho_\alpha(q, q; \beta) &(33)\\ &= \sum_{\alpha \in \pi_1(M_N^2)} \iint_{q_\alpha(0)=q}^{q_\alpha(\hbar\beta)=q} e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \left[\mathcal{H}(q_\alpha(\tau), \dot{q_\alpha}(\tau)) + i\hbar\nu \sum_{i,j} \frac{d\varphi_{ij}^{(\alpha)}(\tau)}{d\tau}\right]} \mathcal{D}q_\alpha \ . \end{aligned}$$

This expression tells us that instead of dealing with anyons governed with the Hamiltonian $\mathcal{H}$, we can work with bosons whose dynamics is dictated by the new Hamiltonian $\mathcal{H}' = \mathcal{H} + i\hbar\nu \sum_{i,j} d\varphi_{ij}^{(\alpha)}(\tau)/d\tau$. In particular we could treat fermions with the Hamiltonian $\mathcal{H}$ as bosons with Hamiltonian $\mathcal{H}' = \mathcal{H} + i\hbar \sum_{i,j} d\varphi_{ij}^{(\alpha)}(\tau)/d\tau$.

Now I think that in this case since $\rho(q, q; \beta)$ has to be a real positive function as well as all the $\rho_\alpha(q, q; \beta)$ one has to add the constraints

$$\sum_{\alpha \in \pi_1(M_N^2)} \sin(\nu n_\alpha \pi) \rho_\alpha(q, q; \beta) = 0 \quad , \tag{34}$$

$$\sum_{\alpha \in \pi_1(M_N^2)} \cos(\nu n_\alpha \pi) \rho_\alpha(q, q; \beta) > 0 \quad . \tag{35}$$

Is not clear to me how this constraints would influence the phases.

Conclusions

The configuration space of identical hard core two dimensional particles has a non trivial topology.

- If the particles are free to move in $I\!R^2$ or in a finite $L \times L$ box then the configuration space is infinitely connected (see fig. 3). Its fundamental group is the braid group whose representations are labeled by an arbitrary parameter ν called statistics. This unusual statistics can be implemented on ordinary particles (for instance bosons) by the addition of a topological statistical interaction.

- If the particles are free to move in a finite box with periodic boundary conditions (a torus: compact Riemann surface of genus 1) then only bosons and fermions are possible if the many body wavefuncions carry a one dimensional representation of $B_N(\text{torus}) = S_N$ [11].

Consider the $N = 2$ case. We have seen that when the particles are free to move on all $I\!R^d$ then the center of mass coordinate splits off in a trivial way. Let' s see what we can easily say about the configuration spaces of particles confined in a box (B) or in a periodic box (PB):

[1d-B] Call $x_1 \in [0, L]$ and $x_2 \in [0, l]$ the particles coordinates. In this case (see fig. 12),

$$M_2^1 = \{(x_1, x_2) : \quad x_2 \in [0, L],\ x_2 < x_1 \leq L\} \quad , \tag{36}$$

which is simply connected. So only boson statistics is allowed. We

could have introduced the center of mass coordinate $R = (x_1 + x_2)/2 \in$

[11] If the many body wave function is a multiplet with k components one finds $\nu = p/q$ with p and $q = k$ coprime integers, and consequently the following restriction on the number of particles: $N = qn$ where n is a non negative integer.

$[0, L]$ and the relative coordinate $r = x_2 - x_1$. Using this coordinates $M_2^1 = [0, L] \times r_2^1$ with $r_2^1 =]0, \min(L - 2R, 2R)]$ (see fig. 13). Now [0,L]

and r_2^1 are both simply connected so M_2^1 is simply connected.

[1d-PB] Using the center of mass coordinate $R \in [0, L]$ and the relative coordinate r one sees by inspection that (see fig. 13),

$$M_2^1 = \{(R, r) : \quad R \in [0, L/2],\ 2R - L \leq r \leq 2R, \tag{37}$$

$$(R, 2R) = (R, 2R - L),\ (0, -r) = (L/2, r)\} \tag{38}$$

$$-\{(R, 0)\ \forall R,\ (0, -L),\ (L/2, 0)\} \quad , \tag{39}$$

which is infinitely connected (as shown in fig. 14 two loops with different winding around the missing point $(0, -L) = (L/2, 0)$ are homotopically inequivalent). So anyon statistics is allowed. The same thing can

be seen introducing the center of mass angle ϕ and the relative angle φ (see fig. 15) The rectangle in the (ϕ, φ) plane defined by $0 \leq \phi \leq \pi$

and $0 \leq \varphi \leq 2\pi$ includes all possible configurations, except for the left and right edges where $(0, \varphi)$ and $(\pi, 2\pi - \varphi)$ both represent the same configuration. Because of this identification the rectangle becomes a Möbious band which is still infinitely connected.

[2d-B] Using the same argument used for the [1d-B] we can say that $M_2^2 = ([0, L] \times [0, L]) \times r_2^2$ where r_2^2 is a space with the same topology as the cone without the tip introduced in the case of particles without boundaries. The only difference beeing that the cone now doesn' t extend to infinity but is finite and its "length" depends on R. So once again since r_2^2 is infinitely connected also M_2^2 is. And anyon statistics is allowed.

[2d-PB] In this case I have no easy way of inferring that the configuration space has to be only doubly connected. I can only say that something similar was happening from going from the 2d plane (M infinitely connected) to the 3d space (M doubly connected). Now in 1d-PB M is infinitely connected and in 2d-PB M is doubly connected.