Effect of temperature on the equilibrium of a white dwarf in general relativity

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We study the effects of treating the equilibrium of a white dwarf within general relativity. We approximate the electron gas within the star with an ideal, perfect, degenerate one. We determine the magnitude of the correction on the Chandrasekhar, Newtonian, limiting mass for central densities approaching the neutron drip critical one. We also discuss the effect of considering a temperature gradient within the star interior due to heat transfer. We point out that, whereas the degenerate hypothesis is fully legitimate, dropping the ideal and perfect assumption may lead to observable effects. This would require a careful study of the Jellium ground state on curved spacetime.

Keywords: White dwarf; General relativity; Temperature; Equilibrium; Stability; Hydrodynamics; Perfect fluid; Chandrasekhar mass; Tolman-Oppenheimer-Volkoff equation

CONTENTS

I. INTRODUCTION

A white dwarf has a central mass density $\rho_c < \rho_c^{\rm nd}$ where $\rho_c^{\rm nd} = 4 \times 10^{11}{\rm g/cm}^3$ is the mass density at which the leptons and baryons composing the star undergo the neutron drip process where by inverse β^+ decay one electron and one proton change into a neutron and a neutrino [1]. A process which ultimately will change the white dwarf into a neutron star. Chandrasekhar [2–4] predicted an equilibrium mass for a white dwarf given by $M_{\rm Ch} = 1.45639 M_\odot$ and independent from the central density. His argument is based on Newtonian hydrodynamics and the thermodynamics of an ideal (non interacting), perfect (adiabatic), and degenerate (at zero temperature) electron gas.

First Tolman [5], and later Oppenheimer and Volkoff [6] proposed a way to treat the hydrodynamic equations for the equilibrium and stability of a star within the framework of general relativity. Even if their hydrodynamic equations have been used mainly for the description of neutron stars for which general relativity effects cannot be neglected due to their extreme compactness, nonetheless they may be used also for the description of a white dwarf.

In this work we want to find a lower bound to the central mass density of a white dwarf necessary to observe a correction to the Chandrasekhar result for the equilibrium mass due to the general relativity description of Tolman-Oppenheimer-Volkoff larger than 2%.

In a previous work [7] we studied the effect of a non zero temperature on a white dwarf Newtonian equilibrium and structure. Here we discuss how one could take care of a temperature gradient throughout the star due to heat transport for the star radiation [2] and we conclude that the temperature profile starting from a central temperature rapidly decays to zero on a radial length scale many orders of magnitude smaller then the radial scale of the mass density profile.

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[1] For some more details on the neutronization thresholds of a white dwarf see table 3.1 of the book [1].
[2] For a discussion of the star cooling see chapter 4 of the book [1].

In order to make some progress over the assumptions of an ideal and perfect electron gas it would be necessary to extract a more realistic equation of state. This has been accomplished with various mean field theories well illustrated in section §2 of the book of Shapiro and Teukolsky [1]. Since for a white dwarf the Wigner-Seitz radius is samll, $0.0003 < r_s < 0.01$ [3], the corrections due to the Coulomb interaction will be small. The principal effect of the electrostatic corrections is to give smaller radii and larger central densities compared with Chandrasekhar's models of the same mass. Nonetheless, in order to make further progress towards an even more accurate equation of state many body methods are necessary. This is not simply an academic exercise because the stars are probably the objects more observed and measured in nature and we can hope to better understand the laws of nature by comparing our theories on earth with the data from astronomical observations. Until recently stars were only available in the sky. Only recently we are able to create a star artificially in a earthly laboratory [8]. Moreover we may hope to be able to detect some gravitational wave generated by binary systems of white dwarfs as they spiral closer to an eventual merger. These systems, which can be detached or interact through mass transfer, are major sources of gravitational waves for future detectors like the Laser Interferometer Space Antenna (LISA) planned by the European Space Agency (ESA) [9]. One may then start for example from the properties of Jellium [10–13] where the ions component is approximated by a uniform neutralizing background. This is just a first brute approximation to the more realistic model of a two component plasma [14, 15] [4], but even so it poses the extremely challenging problem of the determination of the ground state properties of a many electron system on a curved spacetime [16, 17] with the additional subtleties of overcoming the fermion sign problem [18–20] and ordering problems on properly self adjoint operators subject to holonomic constraints as the ones necessary in a quantum theory of curved spacetime [21–24].

II. DISCUSSION

In this work we consider *spacetime* as a smooth manifold $\mathscr{M}$ of dimension $d = 4$ and metric tensor $\boldsymbol{g}$ with covariant components $g_{\alpha\beta}$. We will denote with an arrow over a bold face letter the corresponding 4 vector and with just the bold face symbol the corresponding 3 dimensional vector. Greek indexes run over the d spacetime dimensions. Roman indexes run only over the $d-1$ space dimensions. We use Einstein summation convention of tacitly assuming a sum over repeated indexes. We will use geometrized units $c = G = 1$ throughout.

In Ref. [7] we determined how the Chandrasekhar argument for the limiting mass of a white dwarf at zero temperature could be modified to take into account the effects of a finite non zero temperature. In studying a *spherical* and *static* star equilibrium one introduces the mass $m(r)$ inside a radius r, the mass energy density $\rho(r)$, the pressure $p(r)$, and the temperature $T(r)$ at a radius r.

In Ref. [25] we showed that in general relativity hydrodynamics, from the divergenceless of the stress energy tensor of a *perfect fluid*, i.e. a fluid with a stress energy tensor given by $T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + pg^{\alpha\beta}$ in its isotropic frame, follows, in a local Lorentz frame,

$$\frac{d\rho}{d\tau} + (\rho + p)\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}} = 0, \tag{2.1}$$

$$(\rho + p)\vec{\boldsymbol{a}} = -\vec{\boldsymbol{\nabla}}p - \vec{\boldsymbol{u}}\frac{dp}{d\tau}, \tag{2.2}$$

where ρ is the mass energy density, p is the pressure, τ is the fluid proper time, $\vec{\boldsymbol{u}} = (\gamma, \gamma\boldsymbol{v})$ is the fluid 4 velocity, where $u^0 = dt/d\tau = \gamma = (1 - v^2)^{-1/2}$ is Lorentz factor, and $\vec{\boldsymbol{a}} = d\vec{\boldsymbol{u}}/d\tau$ is the 4 acceleration. The first Eq. (2.1) is the *continuity equation* and the second (2.2) is the *Euler equation* (see also sections §22.3 and §23.5 of the book [26]).

In §6.10 of the book [1] the effects of general relativity on the Newtonian Chandrasekhar argument are determined. They show that for white dwarfs it is often enough to approximate the general Euler equation (2.2) with its non relativistic and Newtonian limits. This does not appreciably alter the picture that includes general relativity effects on the star *equilibrium*. They find that general relativity affects only the *stability* of a white dwarf (see their (6.10.28) equation). Here we will show that albeit very small the general relativity effect on dense white dwarfs equilibrium is still appreciable. Moreover we will discuss the effect of taking care of a finite non zero temperature which, instead of being treated as uniform throughout the whole star, may in general be considered as a function of the distance r from the star center, to be integrated out together with the mass, the mass density, and the pressure.

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[3] The lower bound is dictated by the requirement of being below neutron drip, i.e. $\rho_c < \rho_c^{\rm nd}$. The upper bound can be inferred from figure 3.2 of the book [1] taking a star mass as low as $0.7M_\odot$

[4] Note that since the mass of a proton is about 1000 electron masses the ions component diffusion would be 1000 times slower making it much more classical in a first principles statistical physics description.

In their book Shapiro and Teukolsky [1] treat the uniform zero temperature case. They start [27] from the non relativistic version of (2.1) and (2.2), namely

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}(\rho\boldsymbol{v})=0, \tag{2.3}$$

$$\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}=-\frac{1}{\rho}\boldsymbol{\nabla}p, \tag{2.4}$$

which are obtained from (2.1) and (2.2) in the non relativistic limit $v\ll 1$, $\gamma\approx 1$, and, since the thermal energy is much smaller than the rest mass of the fluid, $p\ll\rho$. And in the left hand side of Euler equation they assume the Newtonian result

$$\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}=\frac{m}{r^2}\hat{\boldsymbol{r}}, \tag{2.5}$$

where m is the mass of the star inside a radius r. They suggest that working within these approximations to the full general relativity picture is enough to extract the main general relativity effects on the stability of the star (see their (5.10.28) equation). The equilibrium properties of the white dwarf being unaltered from the Chandrasekhar analysis (see their (6.10.26) equation). In the following we will only worry about the equilibrium properties of a white dwarf and we want to estimate the extent of the general relativity corrections to the Chandrasekhar original Newtonian analysis.

An important result of Newtonian gravitation is that at any point outside a *spherical* mass distribution, the gravitational field depends only on the mass interior to that point. Moreover, even if the mass interior is moving spherically symmetric, the field outside is constant in time. This result is also true in general relativity, where it is known as *Birkhoff theorem*: the only vacuum, spherically symmetric gravitational field is *static*. The solution of Einstein field equations outside a spherically symmetric mass m, $G^{\alpha\beta}=0$, is the Schwarzschild metric $ds^2=-f(r)\,dt^2+f^{-1}(r)\,dr^2+r^2\,d\Omega^2$, with Ω the solid angle and $f(r)=1-2m/r$ [5]. Inside the star, Einstein field equations become $G^{\alpha\beta}=8\pi T^{\alpha\beta}$. In this case one can show [28] that the metric is given by $ds^2=-h(r)\,dt^2+f^{-1}(r)\,dr^2+r^2\,d\Omega^2$, where now $m=m(r)$ is the mass of the star inside a radius r. In a comoving reference frame the stress energy tensor is diagonal with $T^i{}_i=p$ and $T^0{}_0=-\rho$. Since the stress energy tensor is divergenceless $0=\boldsymbol{\nabla}_\alpha T^\alpha{}_r=\partial_r T^r{}_r+\Gamma^\alpha{}_{r\alpha}T^r{}_r-\Gamma^\alpha{}_{r\beta}T^\beta{}_\alpha$ with Γ's the Christoffel symbols for the metric inside the star [6], it follows

$$(\ln h)'=-\frac{2}{\rho}\frac{dp}{dr}\left(1+\frac{p}{\rho}\right)^{-1}, \tag{2.6}$$

where the prime stands for a derivative respect to r. The equivalence principle and the static assumption allow to rewrite Euler Eq. (2.2) as follows

$$(\rho+p)\frac{D\vec{\boldsymbol{u}}}{d\tau}=-\vec{\boldsymbol{\nabla}}p, \tag{2.7}$$

where by the static assumption $dp/d\tau=0$ has been used and $D\dots/d\tau$ is now a covariant derivative. Then $D\vec{\boldsymbol{u}}/d\tau=D(u^\alpha\vec{\boldsymbol{e}}_\alpha)/d\tau=\vec{\boldsymbol{e}}_\alpha du^\alpha/d\tau+\Gamma^\gamma{}_{\alpha\beta}u^\alpha u^\beta\vec{\boldsymbol{e}}_\gamma$. Multiplying Eq. (2.7) by $\vec{\boldsymbol{e}}_r$, recalling that $\vec{\boldsymbol{e}}_r\cdot\vec{\boldsymbol{e}}_r=g_{rr}=1/f$, using the proper $\Gamma^r{}_{00}$, and choosing a comoving reference frame where $\vec{\boldsymbol{u}}=(u^0,\boldsymbol{0})$, we find $dp/dr=-\rho(1+p/\rho)\Gamma^r{}_{00}(u^0)^2g_{rr}$ where $(u^0)^2=(dt/d\tau)^2=1/h$. As expected, this is an identity. In order to make some progress towards a relativistic structure equation we use the radial component of the Einstein field equations, namely $G^r{}_r=8\pi T^r{}_r=8\pi p$. Keeping in mind that $G^r{}_r=f[(\ln h)'r-1+f]/r^2$ and using Eq. (2.6) (or (2.7)) one can solve for the radial derivative of the pressure to find the *Tolman-Oppenheimer-Volkoff* (TOV) equation [5, 6]

$$\frac{dp}{dr}=-\frac{\rho m}{r^2}\left(1+\frac{p}{\rho}\right)\left(1+\frac{4\pi r^3p}{m}\right)\left(1-\frac{2m}{r}\right)^{-1}, \tag{2.8}$$

where, taking care of the proper volume element of the Schwarzschild space $4\pi r^2\sqrt{g_{rr}}$, the mass $m(r)$ must satisfy

$$\frac{dm}{dr}=4\pi r^2\rho\left(1-\frac{2m}{r}\right)^{-1/2}, \tag{2.9}$$

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[5] The only non zero Christoffel symbols are: $\Gamma^0{}_{0r}=(\ln f)'/2$, $\Gamma^r{}_{00}=(\ln f)'f^2/2$, $\Gamma^r{}_{rr}=-(\ln f)'/2$, $\Gamma^r{}_{\theta\theta}=-rf$, $\Gamma^r{}_{\phi\phi}=-rf\sin^2\theta$, $\Gamma^\theta{}_{r\theta}=1/r$, $\Gamma^\theta{}_{\phi\phi}=-\sin\theta\cos\theta$, $\Gamma^\phi{}_{r\phi}=1/r$, $\Gamma^\phi{}_{\theta\phi}=\cot\theta$, where the prime indicates a radial derivative.

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with boundary condition $m(r=0)=0$. The Newtonian limit is recovered by taking $p \ll \rho$ and $m \ll r$ in Eqs. (2.8) and (2.9). A star made of a perfect fluid must also satisfies an adiabatic equation of state

$$p = K\rho^{\Gamma}, \tag{2.10}$$

known as a *polytrope* of polytropic index $1/(\Gamma-1)$. As shown in Ref. [7] $\Gamma = d\ln p/d\ln\rho \to 4/3$ at high electron mass density, which corresponds to the *extremely relativistic regime* $v \approx 1$, i.e. when we can use a dispersion relation $\epsilon_{\rm er}(k) = \hbar ck$ for the electrons. Assuming that the white dwarf is made up of a completely ionized plasma then we may write $\rho = \mu_e m_u n$ with n the electrons number density, m_u the atomic mass unit, and $\mu_e = A/Z$ with Z the atomic number and A the mass number, so that for example for a ^{4}He or a ^{12}C white dwarf $\mu_e = 2$ and for a ^{56}Fe white dwarf $\mu_e = 56/26 \approx 2.134$. In the extremely relativistic regime K in the polytrope Eq. (2.10) is given by

$$K = \frac{\pi^{2/3}\hbar}{g^{1/3}(\mu_e m_u)^{4/3}}\frac{f_4(z)}{f_3^{4/3}(z)}, \tag{2.11}$$

where, as shown in §61 of Ref. [29], $g=2$ is the electrons spin degeneracy and $f_\mu(z) = -\sum_{\nu=1}^{\infty}(-z)^\nu/\nu^\mu$ for an ideal Fermi-Dirac gas with an *activity* $z = \exp(\beta\mu)$, temperature $T = 1/k_B\beta$, with k_B Boltzmann constant, and μ the chemical potential of the electrons. Note that at $T \to 0$, $z \to \infty$ and in Eq. (2.11) we find $\lim_{z\to\infty} f_4(z)/f_3^{4/3}(z) = 3^{1/3}/2^{5/3}$. In Fig. 1 we show the polytrope exponent $\Gamma = d\ln p/d\ln n$ for the equation of state of the *fully relativistic* [7] electron gas as a function of density at zero temperature. This ideal fully relativistic degenerate electron gas adiabatic is used for example to determine the dashed lines in figures 3.1 and 3.2 in the book [1]. This should be compared with Fig. 1 of Ref. [7] where the same plot is presented but now along an adiabatic equation of state at finite temperature.

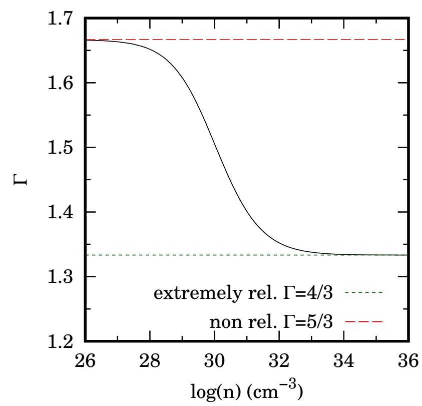

FIG. 1. We show $\Gamma = d\ln p/d\ln n$ for the equation of state of the fully relativistic ($\epsilon_{\rm fr}(k) = \sqrt{(\hbar k)^2 + m_e^2}$) ideal electron gas as a function of density at zero temperature $z \to \infty$. $g = 2$ and n is in cm^{-3}. The gas is in a non relativistic ("non rel." $\epsilon_{\rm nr}(k) = (\hbar k)^2/2m_e$) regime for n as large as approximately 10^{28}cm^{-3}. Whereas it enters the extremely relativistic ("extremely rel." $\epsilon_{\rm er}(k) = \hbar k$) regime for n approximately greater than 10^{31}cm^{-3}. Compare with Fig. 2.3 of Ref. [1].

Let us denote with $p(r) = \bar{p}(\rho(r), T(r))$ the equation of state for each shell between r and $r + dr$ of the star. For the polytrope of Eq. (2.10) $T(r) = 0$ for any r and $\bar{p} = K\rho^\Gamma$. In order to integrate the Euler equation of TOV (2.8) one chooses $m(r=0) = 0$, a central mass density $\rho(r=0) = \rho_c$ and pressure $p(r=0) = \bar{p}(\rho_c, 0)$ and integrates it with Eqs. (2.9) and (2.10) till $r = R$ where $p(r=R) = 0$ and $m(r=R) = M$, with R, M respectively the radius and mass of the star.

We integrated numerically TOV equation and its Newtonian limit (N) in order to understand when the general relativity effects become important. In the N limit this reduces to the *Lane-Emden* problem as discussed in [7].

At First we chose a high enough central electron number density $n_c = 10^{36}$cm^{-3} with $\rho_c = \mu_e m_u n_c$ and a central pressure $p_c = K\rho_c^\Gamma$ with $\Gamma = 4/3$. Moreover, following Chandrasekhar, we chose the case of degenerate electrons at $T \to 0$, i.e. $z \to \infty$ in Eq. (2.11). At this high central number density the electron gas adiabatic equation of state is, to a good approximation, a polytrope in the extreme relativistic $\Gamma \to 4/3$ limit. Our results are shown in Fig. 2.

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[7] Assuming an electron dispersion relation $\epsilon_{\rm fr}(k) = \sqrt{(\hbar k)^2 + m_e^2}$. Note that in the extremely relativistic case, unlike the fully relativistic case, the entropy per electron is just a function of the activity so on an adiabatic the activity is fixed.

From the figure we see that the effects of general relativity are indeed non negligible. In the Newtonian case (N) we recover the Chandrasekhar result of a limiting mass independent from the central density, whereas the TOV equation, unlike the Lane-Emden equation, still has a limiting mass but it depends on the central density. For the existence of the limiting mass in both the TOV and its N limit it is crucial to choose a polytrope equation of state with index less than 5, i.e. $\Gamma > 6/5$. Nonetheless we see from our figure that in order to be able to observe some general relativity effects we had to have a central mass density $\rho_c \approx 3 \times 10^{12}\text{g/cm}^3$ which is one order of magnitude above the neutron drip density of $4 \times 10^{11}\text{g/cm}^3$.

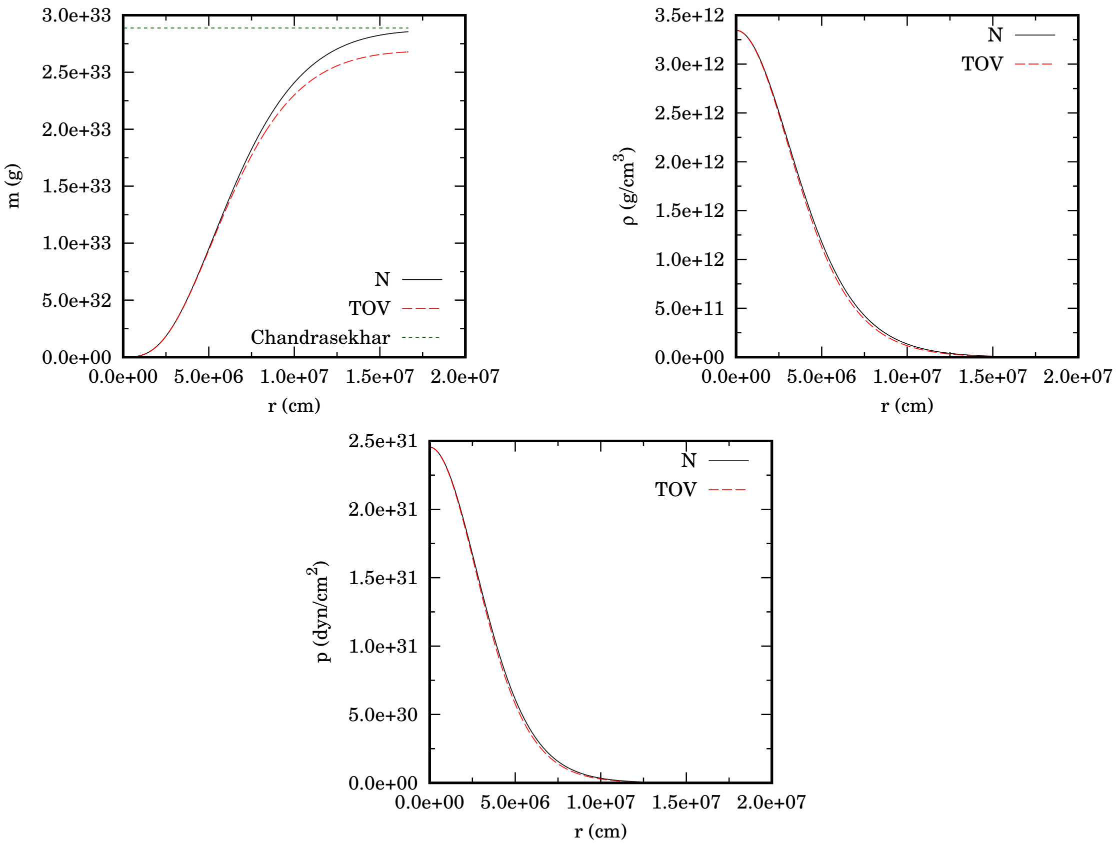

FIG. 2. We compare the numerical integration of the TOV equilibrium equation (TOV) with its Newtonian limit (N) for a white dwarf with $\mu_e = 2$ starting from a central electron number density $n_c = 10^{36}\text{cm}^{-3}$ and a central pressure $p_c = K\rho_c^\Gamma$ with $\rho_c = \mu_e m_u n_c$ at zero temperature $z \to \infty$. In the first panel we show the mass of the star, in the second the mass density, and in the third the pressure. In the first panel we also show the Chandrasekhar limiting value for the star mass, $M_{\rm Ch} = 2.018[4\pi(K/\pi)^{3/2}] = 1.456M_\odot$, the Chandrasekhar radius being $R_{\rm Ch} = 6.897[K/\pi\rho_c^{2/3}]^{1/2} \propto \rho_c^{-1/3}$. As one can see from the second panel the central mass density is an order of magnitude higher than the density for neutron drip. So the white dwarf would very likely turn into a neutron star starting from in its core.

We therefore chose a central density $\rho_c = 10^{11}\text{g/cm}^3$, below neutron drip, with a central electron number density $n_c = \rho_c/(\mu_e m_u) \approx 10^{34}\text{cm}^{-3}$ for which we have a central Fermi wave vector $k_{\rm F} = (6\pi^2 n_c/g)^{1/3}$ and a central Fermi temperature of $T_{\rm F} = E_{\rm F}/k_B = \sqrt{(\hbar k_{\rm F})^2 + m_e^2}/k_B \approx 2 \times 10^{11}\text{K}$, where m_e is the mass of the electron. Therefore a typical white dwarf temperature is much less than $T_{\rm F}$ and the electron gas can then be considered degenerate to a good approximation. From Fig. 1 (or Fig. 1 of Ref. [7]) we see that in this conditions we are well within the extremely relativistic $\Gamma \to 4/3$ regime. Choosing then again $z \to \infty$ in Eq. (2.11) and a central pressure $p_c = K\rho_c^{4/3}$ we observe the results of Fig. 3 which still shows a non negligible effect of general relativity on the white dwarf equilibrium.

As a lower bound $\rho_c^{\rm lb} < \rho_c < \rho_c^{\rm nd}$ to the central mass density of a white dwarf necessary to detect an equilibrium mass of the star such that $(M - M_{\rm Ch})/M_{\rm Ch} > 2\%$ due to general relativity effects we found $\rho_c^{\rm lb} \approx 1.25 \times 10^{11}\text{g/cm}$ when $M \approx 2.756 \times 10^{33}\text{g} \approx 1.386M_\odot$ and $R \approx 4.113 \times 10^7\text{cm} \approx 6 \times 10^{-4}R_\odot$. at the neutron drip mass density the general relativity effects on the equilibrium mass are $\approx 3\%$.

As discussed in Ref. [7], at finite non zero temperature the adiabatic equation of state of the ideal Fermi gas of electrons will enter the extremely relativistic regime at high electron number density. In Fig. 1 of that reference it

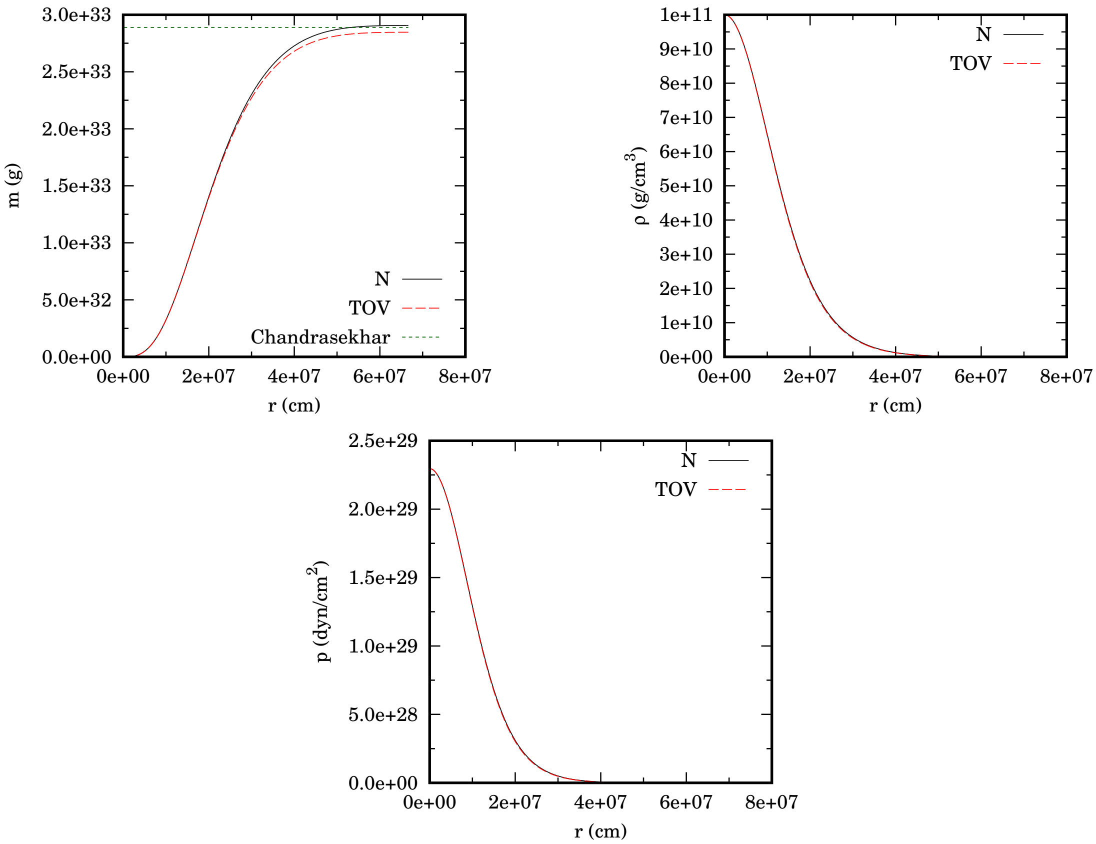

FIG. 3. We compare the numerical integration of the TOV equilibrium equation (TOV) with its Newtonian limit (N) for a white dwarf with $\mu_e = 2$ starting from a central mass density $\rho_c = 10^{11}\mathrm{g/cm}^3$, below neutron drip, and a central pressure $p_c = K\rho_c^{4/3}$ at $z \to \infty$. In the first panel we show the mass of the star, in the second the mass density, and in the third the pressure. In the first panel we also show the Chandrasekhar limiting value for the star mass. As we can see comparing the first panels of the previous Fig. 2 the limiting mass from the TOV equation does depend on the central density. From this first panel we also see that the effect of general relativity is still not negligible.

was shown for example that at a temperature of $T = 20000\mathrm{K}$ the adiabatic equation of state becomes a polytrope with $\Gamma \approx 4/3$ already for $n \gtrsim 10^{25}\mathrm{cm}^{-3}$. In order to take properly into account the effect of temperature on the star equilibrium we would need to introduce a *temperature profile* $T = T(r)$ dependent on the radial distance r within the star. The assumption of Chandrasekhar was to consider negligible the effect of temperature since the average temperatures of a white dwarf $100000 - 3000\mathrm{K}$ are much smaller than the Fermi temperature and the Fermi-Dirac distribution function can be considered sharp. The electron gas will be in its polytrope extremely relativistic regime in the inner star shells and it will remain in this regime in the outer shells where the number density approaches zero where $T_{\mathrm{F}} = m_e/k_B \approx 6 \times 10^9\mathrm{K}$.

The process of integrating over temperature as a function of radial distance is part of solving the equation of energy transport, which is one of the four fundamental equations of stellar structure

$$\frac{dT}{dr} = -\frac{1}{\kappa}\frac{L}{4\pi r^2}\left(1 - \frac{2m}{r}\right)^{1/2}, \tag{2.12}$$

where for a ^{4}He white dwarf $\kappa \approx 10^4 - 10^7\mathrm{erg(s\ cm\ K)}^{-1}$ is the thermal conductivity, and $L \approx 10^{-2} - 10^{-5}L_\odot$ is the average luminosity [30] [8]. This equation determines the temperature gradient necessary to carry the star internal

[8] In a white dwarf, where nuclear reactions do not occur, no energy is generated, so the luminosity is constant within the star. For example early estimates of the luminosity of Sirius B estimated from the observed flux and known distance set his luminosity to about 1/360 of that of the sun. In 1914 W. S. Adams [31] by assigning an effective temperature of 8000K to Sirius B from these spectral measurements and using the equation for black body emission, $L = 4\pi R^2\sigma T_{\mathrm{eff}}^4$, a radius R of $18.8 \times 10^8\mathrm{cm}$ could be inferred for the star (this is just about four times bigger than the modern value). Here we are neglecting pycnonuclear reactions whose effects are described in section §3.7 of the book [1].

luminosity outwards. It should be integrated together with the TOV equation (2.8), the mass equation (2.9), and the adiabatic equation of state. For an ideal fully relativistic electron gas (see the appendix of Ref. [7]), this is given in parametric form by

$$\beta p = g \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{\ln z - \beta\epsilon_{\rm fr}(k)}{e^{\beta\epsilon_{\rm fr}(k)}/z + 1} + \frac{s\rho}{\mu_e m_u}, \tag{2.13}$$

$$\rho = g\mu_e m_u \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{1}{e^{\beta\epsilon_{\rm fr}(k)}/z + 1}, \tag{2.14}$$

where the activity z is the parameter, $k_B s$ is the constant entropy per electron, and $\epsilon_{\rm fr}(k) = \sqrt{(\hbar k)^2 + m_e^2}$ is the electron dispersion relation [9].

In order to determine the temperature profile within the star one needs to integrate numerically the TOV equation (2.8), the mass equation (2.9), the temperature equation (2.12), and the equation of state (2.13)-(2.14), choosing as boundary condition $m(r = 0) = 0$, a central mass density $\rho(r = 0) = \rho_c$, a central temperature $T(r = 0) = T_c$, and a central pressure $p(r = 0) = \bar{p}(\rho_c, T_c)$. But we soon realize that even if the core of the star is at a temperature T_c in the white dwarf range it will rapidly drop to zero at a radius of the order of $\Delta r \approx T_c 4\pi\kappa/L \ll R$ according to the energy transport Eq. (2.12). Much more rapidly than the drop to zero of the mass density $d\rho/dr = (dp/dr)/K\Gamma\rho^{\Gamma-1}$ with dp/dr of the TOV equation (2.8). Note that at the center dm/dr, $d\rho/dr$, and dp/dr start from zero whereas dT/dr diverges to $-\infty$ so it has a cusp. For example for $T_c = 100000$K, $L = 10^{-2} L_\odot$, and $\kappa = 10^4$erg(s cm K)$^{-1}$ we find $\Delta r \approx 10^{-22}$cm!

III. CONCLUSIONS

We revisited the problem of the equilibrium of a white dwarf and determined the effect of general relativity on the limiting mass. In particular we saw that using the Tolman-Oppenheimer-Volkoff hydrodynamic equations, instead of the Lane-Emden equation, we have an equilibrium mass which depends on the central density. Moreover we determined that the star equilibrium mass deviations, respect to the Newtonian limiting case, reach up to 3% at neutron drip central densities. The effect of general relativity on the stability of the star was determined in section §6.10 of the book of Shapiro and Teukolsky [1].

We also discussed the effect of considering a temperature gradient throughout the star interior due to heat transfer occurring in the white dwarf radiation process. We concluded that the temperature profile will drop to zero much more rapidly than the mass density profile moving from the center of the star outwards. We found that the temperature drops to zero on a radial scale 29 orders of magnitudes smaller than the star radius.

We plan to drop the assumption of an ideal, non interacting, electron gas in order to be able to find a more accurate equation of state in the future taking care of the Coulomb interaction. Some progress in this direction has already been made [25] which requires path integrals, or more simply ground state statistical physics methods, on curved spacetimes. We propose to use a diffusion Monte Carlo method [20, 32] taking care of some rather subtle issues that may arise in quantum gravity [21]. Determining the general relativity corrections to the calculation carried out in Ref. [18].

AUTHOR DECLARATIONS

Conflicts of interest

None declared.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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[9] In the appendix of Ref. [7] it was shown that in the extremely relativistic regime the entropy is only a function of the activity z so on an adiabatic one must have z constant. In this regime the pressure is a homogeneous function of degree 4 in T and μ and the number density is homogeneous of degree 3 and one sees that the adiabatic equation of state $\bar{p}$ of Eqs. (2.13)-(2.14) reduces to the polytrope with index 3 of Eq. (2.10). Moreover at the densities of interest $z \to \infty$.

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