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Article

Many Body in General Relativity: A Thermal Equivalence Principle

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Abstract

In this paper, we review the physics of many bodies in the context of general relativity. Starting from the stress–energy tensor for one body and moving onto those for a swarm of bodies and for a perfect fluid, we review the relativistic hydrodynamics, kinetic theory, and statistical physics of N identical bodies. We conclude our excursion with a *thermal equivalence principle* in physics.

Keywords: particle; swarm; perfect fluid; hydrodynamics; kinetic theory; statistical mechanics; general relativity; equivalence principle

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1. Introduction

An interesting problem in physics is to study the properties of a (quantum) *many-body* system at a low (non-zero) temperature on a *curved surface*. For example, colloidal particles may be adsorbed or confined on a substrate with non-zero curvature, be it the wall of a porous material, a membrane, a vesicle, a micelle (for example, made of ampiphilic surfactant molecules such as lipids), a biological membrane, the surface of a large solid particle, or an interface in an oil–water emulsion [1]. For a fluid of ^{4}He atoms, it would be interesting to study the superfluidity. For a fluid of electrons, it would be interesting to study the superconductivity.

One important point to discuss is whether the space in which the particles live is exactly two-dimensional, as in the satirical novella of Edwin Abbott Abbott [2], or if it can be treated as *quasi*-two-dimensional. There is a profound difference between the two scenarios to the point that the form of the interaction between the particles also changes. For example, for colloidal particles, one may choose the polarizable hard sphere pair interaction, or for the fluid of helium atoms one may use the Lennard-Jones pair potential, but the distance between the two interacting particles may be chosen either as the geodesic distance between them or the Euclidean distance in the three-dimensional space where the surface is embedded. For the electron gas, the Coulomb pair potential as a solution to the Poisson equation has different forms in two or three dimensions and in general depends on the metric of the curved surface [3].

More generally, it would be desirable to advance our knowledge for treating a quantum many-body system in a Riemannian space. For example, this would allow for possible refinements of the equations of the states of stars, like for example a White Dwarf, which, in our universe, is ruled by general relativity laws and described in terms of a curved spacetime.

These properties can be studied exactly with the path-integral (Monte Carlo) method, and these studies certainly enrich the knowledge on many bodies in (quantum) general

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relativity [4–7]. Not even the two-body problem can be treated analytically in general relativity [8]. The problem of gravitating many bodies should be separated by the problem of many bodies with non-gravitational interactions in general relativity. In fact, mass curves spacetime through the Einstein field equations and gravitating bodies will behave as free particles on that curved spacetime, whereas non-gravitational interactions produce particle accelerations on the spacetime. So, being able to treat many (quantum) bodies on a curved surface would be an important step forward for the much more complicated problem of gravitating many (quantum) bodies in general relativity.

We find it to be of fundamental importance to create a bridge between the two scientific communities of the exact simulations of a many-body (quantum) system and of general relativity. We foresee important progress in the physics of (quantum) gravitating many-body systems beyond the simple ideal gases or hydrodynamic systems that are usually treated [9,10]. We therefore review the physics of many bodies in the context of general relativity. Starting from the stress–energy tensor for one body and moving on to a swarm of bodies, and a perfect fluid, we review the relativistic hydrodynamics, kinetic theory, and statistical physics of N *identical* bodies. We conclude our excursion with a *thermal equivalence principle* in physics that proves the consistency between statistical physics and general relativity.

In this work, we consider *spacetime* as a smooth manifold $\mathscr{M}$ of dimension d and metric tensor $\boldsymbol{g}$ with covariant components $g_{\alpha\beta}$. We will denote the corresponding 4-vector with an arrow over a bold-face letter and the corresponding 3-dimensional vector with just the bold-face symbol. Greek indexes run over the d spacetime dimensions. Roman indexes run only over the $d-1$ space dimensions. We use the Einstein summation convention of tacitly assuming a sum over repeated indexes. We will assume the speed of light $c=1$ throughout.

The paper is organized as follows. In the first three sections, we review well-known concepts in general relativity. In Section 2, we give the definition of the stress–energy tensor for a free particle, a swarm of free particles, and a perfect fluid; in Section 3, we describe the laws of Newtonian and Relativistic hydrodynamics; in Section 4, we describe the kinetic theory approach in general relativity. In the last Section (5), we propose a statistical physics approach in general relativity demonstrating how the two theories of statistical (quantum) theory and general relativity are consistent among themselves. This last section constitutes the main novel result of the work presenting a new thermal equivalence principle.

2. The Stress–Energy Tensor for Free Particles

Here we review the basic definition of the stress–energy tensor of general relativity for a free particle, a swarm of free particles, and a perfect fluid.

2.1. One Particle

For *one body* of mass m we have a self-gravitating system with a stress–energy tensor given by

$$T^{\alpha\beta}(\vec{\boldsymbol{x}}) = m\int u^\alpha u^\beta \delta^{(4)}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{z}}(\tau))\,d\tau, \tag{1}$$

where τ is the body proper time, $d\vec{\boldsymbol{z}}/d\tau = \vec{\boldsymbol{u}} = (\gamma, \gamma\boldsymbol{v})$ with $u^0 = dt/d\tau = \gamma = (1-v^2)^{-1/2}$, and

$$T^{\alpha\beta}(\vec{\boldsymbol{x}}) = m\frac{u^\alpha u^\beta}{u^0}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}(t)), \tag{2}$$

where the body is at $\boldsymbol{z}(t)$ with velocity $\boldsymbol{v}(t)$ at time t.

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2.2. Swarm of Particles

For a *swarm* of N bodies all of the same mass m and $\boldsymbol{v}$

$$\begin{aligned} T^{\alpha\beta}(\vec{\boldsymbol{x}}) &= mu^{\alpha}u^{\beta}\sum_{i=1}^{N}\int \delta^{(4)}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{z}}_i(\tau_i))\,d\tau_i \\ &= m\frac{u^{\alpha}u^{\beta}}{u^0}\sum_{i=1}^{N}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}_i(t)) \\ &= mu^{\alpha}u^{\beta}n, \end{aligned} \tag{3}$$

where

$$n = \frac{1}{u^0}\sum_{i=1}^{N}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}_i(t)), \tag{4}$$

is the proper number density of bodies measured in a *comoving frame* where $\vec{\boldsymbol{u}} = (1, \boldsymbol{0})$.

2.3. Perfect Fluid

For a *perfect fluid* of the proper number density n of non-interacting bodies all of the same mass m and $v = |\boldsymbol{v}|$ but an isotropic velocity profile $\boldsymbol{v} = v\boldsymbol{n}$

$$T^{\alpha\beta} = \chi\langle u^{\alpha}u^{\beta}\rangle_{\boldsymbol{n}}, \tag{5}$$

$T^{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $T^{00} = \chi\gamma^2$. Since $T^{00} = \rho = n(\gamma m)$ is the energy density of the fluid, we require $\chi = mn/\gamma$. Then,

$$\begin{aligned} T^{ij} &= \chi\gamma^2 v^2\langle n^i n^j\rangle_{\boldsymbol{n}} \\ &= \chi\gamma^2 v^2\frac{1}{3}\delta^{ij} \\ &= n(\gamma m)v^2\frac{1}{3}\delta^{ij} \\ &= p\delta^{ij}, \end{aligned} \tag{6}$$

where δ is a Kronecker delta, and in the second equality we used the isotropy of $\boldsymbol{n}$;

$$\begin{cases} \rho = n(m\gamma) \\ p = \frac{1}{3}\rho v^2 \end{cases} \tag{7}$$

are respectively the mass density and pressure in the *isotropic frame* of the fluid. In summary,

$$T^{\alpha\beta} = (\rho + p)u^{\alpha}u^{\beta} + p\eta^{\alpha\beta}, \tag{8}$$

where $||\eta^{\alpha\beta}|| = \mathrm{diag}\{-1, 1, 1, 1\}$ is the metric in Minkowski spacetime. For photons, $v = 1$ and $p = \rho/3$. For $v \ll 1$, $\rho = nm(1 + v^2/2 + \ldots)$, and $p \approx nmv^2/3 = (2/3)(\rho - nm) = (2/3)\epsilon$, where $\epsilon = (3/2)k_B T$ is the internal energy of a monatomic ideal gas in thermal equilibrium at a temperature T, k_B is the Boltzmann constant, and $p = nk_B T$ is the ideal gas equation of state.

3. Hydrodynamics

Hydrodynamics concerns itself with the study of the motion of fluids (liquids and gases). Since the phenomena considered in fluid dynamics are macroscopic, a fluid is regarded as a continuous medium. Therefore, when we speak of the “point” of a fluid (or of an infinitesimal volume of it) we mean not a single molecule of the fluid but a volume element still containing very many molecules, yet small compared with the volume of the whole fluid.

3.1. Newtonian

A mathematical description of the state of a moving fluid consists in specifying the fluid velocity $\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{x})$ and any two thermodynamic functions pertaining to the fluid, for instance the pressure $p = p(t, \boldsymbol{x})$ and the density $\rho = \rho(t, \boldsymbol{x})$, from which one can determine all other thermodynamic quantities. These five quantities are functions of the coordinates $\boldsymbol{x} = (x, y, z)$ and the time t. Once again, we stress that a point $\boldsymbol{r}$ in space at a given time t refers to a fixed point and not to specific particles of the fluid. From Chapter I of Ref. [11], we find that

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}(\rho \boldsymbol{v}) = 0, \tag{9}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p, \tag{10}$$

$$\frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) s = 0, \tag{11}$$

where the first equation is the *continuity equation*, the second is the *Euler equation*, and the third is the *equation for the adiabatic flow* in which $s = s(t, \boldsymbol{x})$ is the entropy per particle.

From the first law of thermodynamics, it follows that

$$d\epsilon = T\, ds - p\, d(m/\rho), \tag{12}$$

$$\epsilon = \epsilon(\rho, s), \tag{13}$$

$$p = \frac{\rho^2}{m} \frac{\partial \epsilon}{\partial \rho}\bigg|_s, \tag{14}$$

$$T = \frac{\partial \epsilon}{\partial s}\bigg|_\rho, \tag{15}$$

where ϵ is the internal energy per particle. Equations (14) and (15) can be considered as algebraic relations for the right-hand sides of Equations (10) and (11), respectively.

For an ideal gas $\epsilon = \epsilon(T)$ and for a monatomic gas,

$$s = k_B \ln(T^{3/2} m/\rho) + \text{constant}. \tag{16}$$

3.2. Relativistic

We will work in a Local Lorentz Frame (LLF). Recalling that the stress–energy tensor is divergenceless, from the stress–energy tensor of a perfect fluid (8) we find that

$$\begin{aligned} 0 = T^{\alpha\beta}{}_{,\beta} &= (\rho + p)_{,\beta} u^\alpha u^\beta + (\rho + p) u^\alpha{}_{,\beta} u^\beta + (\rho + p) u^\alpha u^\beta{}_{,\beta} + p_{,\beta}\eta^{\alpha\beta} \\ &= \frac{d(\rho + p)}{d\tau} u^\alpha + (\rho + p) a^\alpha + (\rho + p) u^\alpha u^\beta{}_{,\beta} + p_,{}^\alpha, \end{aligned} \tag{17}$$

where the comma stands for a partial derivative. Multiplying by $\vec{\boldsymbol{u}}$ and recalling that $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{a}} = 0$, we find that

$$\frac{d\rho}{d\tau} = -(\rho + p)\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}}, \tag{18}$$

which is the relativistic continuity expression which extends Equation (9).

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To find the extension of the Euler equation, we introduce the projector tensor

$$\begin{aligned} P^{\alpha\beta} &= \eta^{\alpha\beta} + u^\alpha u^\beta \quad \text{for } \vec{u} \text{ timelike} \quad \vec{u}\cdot\vec{u} = -1 \\ P^{\alpha\beta} &= \eta^{\alpha\beta} - n^\alpha n^\beta \quad \text{for } \vec{n} \text{ spacelike} \quad \vec{n}\cdot\vec{n} = +1 \end{aligned}$$

Then

$$0 = P_{\alpha\gamma} T^{\alpha\beta}{}_{,\beta} = (\rho + p) a_\gamma + P_{\alpha\gamma} p_{,}{}^{\alpha}, \tag{19}$$

or

$$(\rho + p)\vec{a} = -\vec{\nabla} p - \vec{u}\frac{dp}{d\tau}, \tag{20}$$

which is the relativistic Euler equation which extends Equation (10).

It is easy to see that in the non-relativistic limit $\vec{u} = (\gamma, \gamma \boldsymbol{v}) \approx (1, \boldsymbol{v})$ with $v \ll 1$ and $p \ll \rho$, Equation (18) reduces to Equation (9) and Equation (20) reduces to Equation (10) [12] (to determine the stability of a star, it is often sufficient to replace Equation (20) with Equation (10), as reported in §6.9 of Ref. [9]).

Let us now discuss the continuity Equation (18). First of all, we observe that the mass density is not conserved, $d\rho/d\tau \neq 0$, but the baryon, lepton, charge, ... numbers are conserved. For example, if we refer to $n = N/V$ as the baryon number density in the remaining frame of the fluid with N baryons in the volume V, N is certainly constant but V will change, so that

$$0 = \frac{dN}{d\tau} = \frac{d(nV)}{d\tau}, \tag{21}$$

but $(dV/d\tau)/V = \vec{\nabla}\vec{u}$ (see Ex. 22.1 in Ref. [10]). So,

$$\begin{aligned} 0 &= \frac{1}{V}\frac{d(nV)}{d\tau} \\ &= \frac{dn}{d\tau} + n\vec{\nabla}\vec{u} \\ &= \vec{u}\cdot\vec{\nabla} n + n\vec{\nabla}\vec{u} \\ &= \vec{\nabla}(n\vec{u}), \end{aligned} \tag{22}$$

where we may define the divergenceless current density as

$$\vec{J} = n\vec{u}. \tag{23}$$

Let us now discuss thermodynamics. The second law tells that $ds/d\tau \geq 0$ where s is the entropy per baryon. The first law becomes

$$d(\rho/n) = -p\, d(1/n) + T\, ds, \tag{24}$$

or

$$d\rho = \frac{\rho + p}{n}\, dn + nT\, ds, \tag{25}$$

which is the relativistic extension of Equation (12). In this equation, the differential d can be substituted either with an exterior derivative $\widetilde{\boldsymbol{d}}$, with a gradient $\vec{\boldsymbol{\nabla}}$, or with a directional derivative $\vec{\boldsymbol{\nabla}}_{\vec{u}} = u^\alpha \partial/\partial x^\alpha = d/d\tau$. Given an equation of state $\rho = \rho(n, s)$, we will obtain

$$p = n \frac{\partial \rho}{\partial n}\bigg|_s - \rho, \tag{26}$$

$$T = \frac{1}{n} \frac{\partial \rho}{\partial s}\bigg|_n, \tag{27}$$

which are the relativistic extensions of Equations (14) and (15).

It is easy to show that a perfect fluid flow is adiabatic. From the relativistic continuity Equations (18) and (22), it follows that

$$\frac{d\rho}{d\tau} = \frac{\rho + p}{n} \frac{dn}{d\tau}. \tag{28}$$

Then, from the relativistic first thermodynamic Equation (25), it follows that

$$\frac{ds}{d\tau} = 0. \tag{29}$$

3.2.1. Shock Wave

Consider a homogeneous, static, perfect fluid. A sound wave in the fluid is an adiabatic perturbation. The speed of sound is

$$v_s^2 = \frac{\partial p}{\partial \rho}\bigg|_s \tag{30}$$

Expand

$$\begin{aligned} \rho &= \rho_0 + \rho_1, \\ p &= p_0 + p_1, \\ n &= n_0 + n_1, \end{aligned}$$

where ρ_0, p_0, and n_0 are constant in space (uniform fluid) and in time (static fluid) and ρ_1, p_1, and n_1 are small perturbations. Taking $\vec{\boldsymbol{u}} = (1, \boldsymbol{v}_1)$ with $v_1 \ll 1$, we find from the continuity Equation (18) that

$$\frac{\partial \rho_1}{\partial t} = -(\rho_0 + p_0) \boldsymbol{\nabla} \boldsymbol{v}_1, \tag{31}$$

and from the spatial part of the Euler Equation (20) we find that

$$(\rho_0 + p_0) \frac{\partial \boldsymbol{v}_1}{\partial t} = -\boldsymbol{\nabla} p_1, \tag{32}$$

where we neglect the last term $\vec{\boldsymbol{u}}\, dp/d\tau = \vec{\boldsymbol{u}}\, u^\alpha \partial p/\partial x^\alpha$ because a second-order infinitesimal and $\partial p_0/\partial t = 0$. Therefore, by combining Equations (31) and (32), we find that

$$\frac{\partial^2 \rho_1}{dt^2} = -(\rho_0 + p_0) \boldsymbol{\nabla} \frac{\partial \boldsymbol{v}_1}{\partial t} = \boldsymbol{\nabla}^2 p_1. \tag{33}$$

In a perfect fluid $p = p(\rho, T)$ so that $p(\rho_0 + \rho_1, T) = p(\rho_0, T) + \partial p(\rho_0, T)/\partial \rho|_s \rho_1 = p_0 + p_1$ with $p_1 = v_c^2 \rho_1$, we finally find that

$$\frac{\partial^2 \rho_1}{dt^2} = v_s^2 \boldsymbol{\nabla}^2 \rho_1, \tag{34}$$

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which is the shock wave equation.

3.2.2. Bernoulli Equation

Consider a steady, adiabatic flow of a perfect fluid. Since in a steady state, $\partial p/\partial t = 0$ from the relativistic Euler Equation (20), it follows that

$$(\rho + p)\frac{du^0}{d\tau} = -u^0\frac{dp}{d\tau}, \tag{35}$$

So

$$\begin{aligned} \frac{d}{d\tau}\left(u^0\frac{\rho+p}{n}\right) &= \frac{du^0}{d\tau}\frac{\rho+p}{n} + u^0\frac{d}{d\tau}\left(\frac{\rho+p}{n}\right) \\ &= -\frac{u^0}{n}\frac{dp}{d\tau} + u^0\frac{d}{d\tau}\left(\frac{\rho+p}{n}\right) \\ &= \frac{u^0}{n}\left[\frac{d\rho}{d\tau} - \frac{\rho+p}{n}\frac{dn}{d\tau}\right] = 0, \end{aligned} \tag{36}$$

where in the second equality we used Equation (35) and in the last equality we used the relativistic first law of thermodynamics (Equation (25)). We then conclude that

$$u^0\frac{\rho+p}{n} \quad \text{is constant along the fluid flow lines.} \tag{37}$$

In the Newtonian limit $\vec{u} = (\gamma, \gamma\boldsymbol{v}) \approx (1, \boldsymbol{v})$ with $v \ll 1$, $u^0 = \gamma = (1-v^2)^{-1/2} \approx 1 + v^2/2$, and $P \ll \rho$. We can take $\rho = \rho_0(1+\pi)$ and $(\rho+p)/n \approx \rho_0(1+\pi+p/\rho_0)/n$, so that

$$\frac{1}{2}v^2 + \pi + p/\rho_0 \quad \text{is constant along the fluid flow lines,} \tag{38}$$

where the sum of the last two terms is the *enthalpy*.

4. Kinetic Theory Approach

The kinetic theory approach is based on a *one-body* distribution function [10,13].

4.1. Distribution Function

We will construct a Lorentz-invariant phase space distribution function as the number density of particles in phase space

$$f = \frac{d\mathcal{N}}{d\boldsymbol{x}\,d\boldsymbol{p}}, \qquad \int f\,d\boldsymbol{x}\,d\boldsymbol{p} = N, \tag{39}$$

for a fluid of N bodies, where $d\boldsymbol{x} = dx^1dx^2dx^3$ and $d\boldsymbol{p} = dp^1dp^2dp^3$, so that f/N can be considered as a probability distribution function. We will now prove that f as defined above is a Lorentz-invariant distribution. We start by defining a proper 3-volume. The 4-volume $d^4\Omega = dx^0\,dx^1\,dx^2\,dx^3$ is invariant under a Lorentz transformation. Dividing by $d\tau$, we find another Lorentz-invariant

$$dV = u^0\,dx^1\,dx^2\,dx^3. \tag{40}$$

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Next, we want to define a 3-volume element in momentum space. The 4-volume $d^4p = dp^0\, dp^1\, dp^2\, dp^3$ is invariant. Since $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$, we will define

$$\begin{aligned} d\Pi &= \int d^4p\, \delta\left(\sqrt{-\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{p}}} - m\right) \\ &= \frac{\sqrt{(p^0)^2 - \boldsymbol{p}^2}}{p^0} dp^1\, dp^2\, dp^3 \\ &= \frac{m}{p^0}\, dp^1\, dp^2\, dp^3. \end{aligned} \tag{41}$$

and

$$dV\, d\Pi = dx^1\, dx^2\, dx^3\; dp^1\, dp^2\, dp^3, \tag{42}$$

which is Lorentz-invariant.

We will now prove conservation of volume in phase space in curved spacetime (see Liouville's theorem in BOX 22.6 of Ref. [10]). Consider a very small bundle of identical particles that move through curved spacetime on a neighboring geodesic. We want to prove that $d(dV\, d\Pi)/d\lambda = 0$, where λ is an affine parameter along the central geodesic of the bundle. Given any function of phase space $g(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})$, if $m \neq 0$ (if $m = 0$, see BOX 22.6 of Ref. [10]) take $\tau = a\lambda + b$ for arbitrary a and b. Then

$$\frac{dg(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})}{d\lambda} = \frac{\partial g}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} + \frac{\partial g}{\partial p^\alpha}\frac{dp^\alpha}{d\lambda}, \tag{43}$$

on a geodesic $dp^\alpha/d\tau = 0$ so that

$$\frac{dg(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})}{d\lambda} = \frac{\partial g}{\partial x^\alpha} p^\alpha \frac{a}{m}, \tag{44}$$

and for $g = dV\, d\Pi$ and $p^\alpha = m dx^\alpha/d\tau$,

$$\frac{dg}{d\lambda} = a\frac{dg}{d\lambda}, \tag{45}$$

for any a, so $dg/d\lambda = 0$. Since $d\mathcal{N}$ and $dV\, d\Pi$ are unchanged, then $f = d\mathcal{N}/dV\, d\Pi$ is also unchanged:

$$\frac{df}{d\lambda} = 0. \tag{46}$$

This equation is at the heart of the collisionless Boltzmann equation and the Vlasov equation. All these approximate theories are valid at sufficiently low density, whereas for the full Boltzmann equation, [13,14] one has

$$\frac{df}{d\lambda} = \left(\frac{\partial f}{\partial \lambda}\right)_{\text{collisions}}. \tag{47}$$

This is the most famous of all kinetic equations and was obtained by Boltzmann more than a century ago.

The phase space probability density of a system in thermodynamic equilibrium at an inverse temperature $\beta = 1/k_B T$ with the k_B Boltzmann constant, is not an explicit function of proper time. We shall use the symbol f_0 to denote the equilibrium probability density.

4.2. Ideal Gas

For an *ideal gas*, i.e., a fluid of non interacting many identical bodies, from §37, 52, and 53 in Ref. [15], we know that, on a comoving frame with $\vec{\boldsymbol{u}} = (1, \mathbf{0})$, we can write

$$f_0(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = \frac{d\mathcal{N}}{dV\,d\Pi} = \frac{g}{h^3}\frac{1}{e^{-\beta(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{u}}+\mu)} - \varepsilon}, \tag{48}$$

where h is the Planck constant, μ is the chemical potential, $g = 2J + 1$ is the spin J degeneracy (2 polarizations for photons), and

$$\varepsilon = \begin{cases} +1 & \text{Bose–Einstein statistics} \\ 0 & \text{Maxwell–Boltzmann statistics} \\ -1 & \text{Fermi–Dirac statistics} \end{cases} \tag{49}$$

where $\varepsilon = 0$ takes care of the statistics for classical *distinguishable* bodies at sufficiently high temperature. At sufficiently low temperature, a different statistical approach must be devised, in which the mean occupation numbers of the various quantum states of bodies are not assumed to be small. The statistics, however, differ according to the type of many-body wave function by which the gas is described. These functions must be either symmetrical or antisymmetrical with respect to the interchange of any pair of particles (see §61 in Ref. [16]). The former case occurs for bodies with integral spin, *bosons* $\varepsilon = +1$, and the latter case for those with half-integral spin, *fermions* $\varepsilon = -1$.

4.3. Moments of the Distribution Function of the Ideal Gas

Next, we can take moments of f_0 with respect to $\vec{\boldsymbol{p}}$

$$\int f_0 p^\mu \, d\Pi = J^\mu, \tag{50}$$

$$\int f_0 p^\mu p^\nu \, d\Pi = T^{\mu\nu}, \tag{51}$$

where $d\Pi = d\boldsymbol{p}/p^0$ with $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$. For $\vec{\boldsymbol{u}} = (1, \mathbf{0})$, from Equations (23) and (8), we must have

$$J^\mu = n u^\mu, \tag{52}$$

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p\eta^{\mu\nu}, \tag{53}$$

so

$$\begin{aligned} n &= -J^\mu u_\mu = -\int f_0 p^\mu u_\mu \, d\Pi = \int f_0 \, d\boldsymbol{p} \\ &= \frac{g}{h^3}\int_0^\infty \frac{4\pi p^2 \, dp}{e^{\beta[\sqrt{\boldsymbol{p}^2+m^2}-\mu]} - \varepsilon}. \end{aligned} \tag{54}$$

Introduce the following change of variables

$$\begin{cases} p = m \sinh\chi \\ \bar{\beta} = m\beta \end{cases} \tag{55}$$

so that from Equation (54) we find

$$n = \frac{4\pi g m^3}{h^3}\int_0^\infty \frac{\sinh^2\chi \, \cosh\chi \, d\chi}{e^{[\bar{\beta}\cosh\chi - \beta\mu]} - \varepsilon}. \tag{56}$$

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For the pressure,

$$\begin{aligned} p &= \frac{1}{3}(u_\mu u_\nu + \eta_{\mu\nu})T^{\mu\nu} \\ &= \frac{1}{3}\int f_0 \boldsymbol{p}^2\, d\Pi \\ &= \frac{1}{3}\int f_0 \boldsymbol{p}^2 \frac{d\boldsymbol{p}}{p^0} \\ &= \frac{4\pi g m^4}{3h^3}\int_0^\infty \frac{\sinh^4\chi\, d\chi}{e^{[\bar\beta\cosh\chi - \beta\mu]} - \varepsilon}. \end{aligned} \tag{57}$$

Also,

$$\begin{aligned} \rho - 3p &= -T^\alpha{}_\alpha = m^2 \int f_0 \frac{d\boldsymbol{p}}{p^0} \\ &= \frac{4\pi g m^4}{h^3}\int_0^\infty \frac{\sinh^2\chi\, d\chi}{e^{[\bar\beta\cosh\chi - \beta\mu]} - \varepsilon}. \end{aligned} \tag{58}$$

4.4. *Maxwell–Boltzmann Statistics ($\varepsilon = 0$) [17]*

From Ref. [18], we learn that

$$K_n(\bar\beta) = \frac{\bar\beta^n}{(2n-1)!!}\int_0^\infty d\chi\, \sinh^{2n}\chi\, e^{-\bar\beta\cosh\chi} \tag{59}$$

$$= \frac{\bar\beta^{n-1}}{(2n-3)!!}\int_0^\infty d\chi\, \sinh^{2n-2}\chi\cosh\chi\, e^{-\bar\beta\cosh\chi}, \tag{60}$$

where K_n is a modified Bessel function of the second kind and in the second equality we performed an integration by parts. The asymptotic behaviors of the modified Bessel function are as follows:

$$K_n(\bar\beta) = \sqrt{\frac{\pi}{2\bar\beta}}e^{-\bar\beta}\left[1 + \frac{4n^2-1}{8\bar\beta} + O(\bar\beta^{-2})\right] \quad \bar\beta \gg 1, \tag{61}$$

$$K_n(\bar\beta) = \frac{(n-1)!}{\bar\beta^n}\left[2^{n-1} - \frac{2^{n-3}\bar\beta^2}{n-1} + O(\bar\beta^3)\right] \quad \bar\beta \ll 1. \tag{62}$$

We then find that

$$n = aK_2(\bar\beta)/\bar\beta, \tag{63}$$

$$p = amK_2(\bar\beta)/\bar\beta^2 = n\, k_B T, \tag{64}$$

$$\rho - 3p = amK_1(\bar\beta)/\bar\beta, \tag{65}$$

where $a = 4\pi g m^3 e^{\beta\mu}/h^3$. Note that the ideal gas equation of state (64) is a relativistic invariant.

For the internal energy per particle we then find that

$$u(T) = \frac{\rho}{n} = m\frac{K_1(\bar\beta)}{K_2(\bar\beta)} + 3k_B T = \begin{cases} m\left[1 + \frac{3}{2}\frac{k_B T}{m} + \dots\right] & \bar\beta \gg 1, \\ 3k_B T & \bar\beta \ll 1, \end{cases} \tag{66}$$

where we used the asymptotic expansions (61) and (62).

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For the ratio of the specific heats $\gamma(T) = c_p/c_v$ we then find that

$$\gamma(T) = \frac{\left.\frac{du}{dT}\right|_p}{\left.\frac{du}{dT}\right|_v} = 1 + \frac{k_B}{\left.\frac{du}{dT}\right|_v} = \begin{cases} 5/3 & \bar{\beta} \gg 1, \\ 4/3 & \bar{\beta} \ll 1. \end{cases} \tag{67}$$

4.5. Quantum Statistics ($\varepsilon = \pm 1$) [19]

In our previous study [19], we performed calculations for the quantum statistics of identical particles which require either a symmetrization (for Bose–Einstein statistics) or antisymmetrization (for Fermi–Dirac statistics) of the free distinguishable particle density matrix. Introducing the fugacity $z = e^{\beta\mu}$, for the Bose–Einstein statistics, we find that

$$\beta p = \frac{gm^2}{2\pi^2\beta\hbar^3}\sum_{\nu=1}^{\infty}\frac{z^\nu}{\nu^2}K_2(\beta m\nu)\,, \tag{68}$$

$$n = \frac{gm^2}{2\pi^2\beta\hbar^3}\sum_{\nu=1}^{\infty}\frac{z^\nu}{\nu}K_2(\beta m\nu)\,, \tag{69}$$

and for the Fermi–Dirac statistics

$$\beta p = \frac{gm^2}{2\pi^2\beta\hbar^3}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}z^\nu}{\nu^2}K_2(\beta m\nu)\,, \tag{70}$$

$$n = \frac{gm^2}{2\pi^2\beta\hbar^3}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}z^\nu}{\nu}K_2(\beta m\nu)\,. \tag{71}$$

In Ref. [19], we also give the results for the non-relativistic limit $\epsilon(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2} \approx \boldsymbol{p}^2/2m$ and the extreme relativistic limit $\epsilon(k) \approx |\boldsymbol{p}|$ of these expressions.

At the zero-temperature limit ($\beta \to \infty$), Equations (70) and (71) reduce as follows (see §2.3 of Ref. [9]):

$$p = \frac{g}{2}\frac{m}{\lambda\!\!\!^{-}{}^3}\phi(x)\,, \tag{72}$$

$$n = \frac{g}{2}\frac{x^3}{3\pi^2\lambda\!\!\!^{-}{}^3}\,, \tag{73}$$

$$\phi(x) = \frac{1}{8\pi^2}\left[x\sqrt{1+x^2}\left(\frac{2}{3}x^2 - 1\right) + \ln\left(x + \sqrt{1+x^2}\right)\right], \tag{74}$$

where $\lambda\!\!\!^{-} = \hbar/m$, with m the electron mass, is the electron Compton wavelength.

5. Statistical Mechanics Approach

The statistical mechanics approach is based on a *many-body* distribution function [10,13].

5.1. Thermal Equilibrium of the Many Bodies

The aim of equilibrium statistical mechanics is to calculate the observable properties of a system of interest either as averages over a phase trajectory (the method of Boltzmann), or as averages over an ensemble of systems, each of which is a replica of the system of interest (the method of Gibbs). In the Gibbs formulation of statistical mechanics, the equilibrium probability distribution for the systems of N identical bodies of the ensemble is described by $\rho_0^{(N)}$, a phase space probability density, in a $6N$-dimensional phase space $\prod_{i=1}^N d\boldsymbol{x}_i\, d\boldsymbol{p}_i = d^N V\, d^N \Pi$ in the classical case or a density matrix in the quantum case. Here, we will only consider the more general quantum case that reduces to the classical case at high temperatures. We will then use [13,20,21] for distinguishable bodies

$$\rho_0^{(N)} = \exp\left(-\int_{\partial\Omega} T^{\mu\nu}\beta_\nu\, dS_\mu\right), \tag{75}$$

$$Z_N = \mathrm{tr}\left(\rho_0^{(N)}\right), \tag{76}$$

where $\partial\Omega$ is a general, arbitrary, spacelike hypersurface bounding the 4-volume Ω and $\vec{\boldsymbol{\beta}}(x)$ is a 4-vector such that $\beta = \sqrt{\beta_\mu\beta^\mu}$ and, as usual, $1/k_B\beta(x) = T(x)$ is the invariant absolute temperature, i.e., the temperature measured by a comoving thermometer. Z_N is the canonical partition function where $\mathrm{tr}(\ldots)$ denotes a trace that requires a path integral in position representation [20], where for bosons, one needs to symmetrize the density matrix for distinguishable bodies over permutations of their $\{\vec{\boldsymbol{r}}_i\}$ positions, and for fermions, one needs to antisymmetrize it. So, $\int T^{00}\, dV = \mathscr{H}$ with $\mathscr{H} = \mathscr{K} + \mathscr{V}$ is the Hamiltonian operator of the fluid, where $\mathscr{K}$ is the kinetic energy operator of the N bodies and $\mathscr{V}$ their potential energy. The covariant form of Equation (75) of the equilibrium statistical operator was first used by Weldon [22] for the Belinfante symmetrized stress–energy tensor.

It is then possible to define the n-body reduced equilibrium distribution functions as

$$f_0^{(n)}(\vec{\boldsymbol{x}}_1,\ldots,\vec{\boldsymbol{x}}_n) = \left\langle \left[\prod_{i=1}^{n}\sum_{j=1}^{N}\delta(\vec{\boldsymbol{x}}_i - \vec{\boldsymbol{r}}_j)\right]_{\mathrm{DP}} \right\rangle \tag{77}$$

$$= \frac{1}{Z_N}\int \left[\prod_{i=1}^{n}\sum_{j=1}^{N}\delta(\vec{\boldsymbol{x}}_i - \vec{\boldsymbol{r}}_j)\right]_{\mathrm{DP}} \rho_0^{(N)}(\{\vec{\boldsymbol{r}}_k\},\{\vec{\boldsymbol{r}}_k\};\beta)\, d^{4N}\vec{\boldsymbol{r}}, \tag{78}$$

where the thermal average of an operator $\mathscr{O}$ is $\langle\mathscr{O}\rangle = \mathrm{tr}\left(\rho_0^{(N)}\mathscr{O}\right)/Z_N$, $d^{4N}\vec{\boldsymbol{r}} = \prod_{i=1}^{N} d\vec{\boldsymbol{r}}_i$, $\rho_0^{(N)}(\{\vec{\boldsymbol{r}}_k\},\{\vec{\boldsymbol{r}}'_k\};\beta)$ is the position representation of the density matrix (75) at an inverse temperature β that results from a path integral [20] with the proper symmetrization or antisymmetrization necessary to reflect the permutational properties of the identical bodies, and the subscript DP means that only the products of Dirac delta functions relative to Different Particles should be considered. Here, $f_0^{(1)} = \int f_0\, d\Pi$, where f_0 is the one-body distribution function mentioned in the previous Section (4).

In a recent project [23], we studied an electron gas at low temperatures, *Jellium*, on the surface of a sphere through the path-integral Monte Carlo method. A unit sphere is *a* surface with a constant positive scalar curvature 2. (Being a manifold of dimension $2 < 3$, it is conformally flat; moreover, in a two-dimensional world it is possible to conceive anyonic statistics [24] for identical but impenetrable bodies. For anyons, unlike bosons and fermions, statistics depend on the entire imaginary time evolution and braiding properties of the path and not just on its initial and final point. The braid group was introduced in 1925 by Emil Artin.) In particular, we noticed that the simulation "speed" of the path diminishes in the neighborhood of the poles. This is a consequence of the *hairy ball theorem*, according to which the Euler class is the obstruction to the tangent planes, and the *tangent bundle* (a particular *fiber bundle*) always has a non-vanishing *fiber*, or hair, for any *section* (in topology, a cross-*section* of a fiber (tangent) bundle space $B \times F$ is a graph over the *base space* B, in this case the sphere; the choice of tangent vector at any point of the sphere is a section of the tangent bundle of the sphere). The theorem was first proven by Henri Poincaré for the sphere in 1885 [25], and extended to higher even dimensions in 1912 by Luitzen Egbertus Jan Brouwer [26]. The theorem has been expressed colloquially as "you can't comb a hairy ball flat without creating a cowlick" or "you can't comb the hair on a coconut". If z is a continuous function that assigns a vector in the three-dimensional space

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to every point $\mathscr{P}$ on a sphere such that $z(\mathscr{P})$ is always tangent to the sphere at $\mathscr{P}$, then there is at least one pole, a point where the field vanishes, i.e., a $\mathscr{P}$ such that $z(\mathscr{P}) = 0$. Every zero of a vector field has a (non-zero) *index* (the index of a bilinear function/al is the dimension of the space on which it is negative-definite; according to the Morse theorem, from the calculus of variations, there is a relation between the conjugate points (a point of the path where it ceases to be a minimum of the action) along a classical path to the negative eigenvalues of $\delta^2 S$, where S is the action in the path integral. More precisely, the Morse index theorem states that, for an extremum $\vec{\boldsymbol{r}}(t), 0 < t < \beta$, the index of $\delta^2 S$ is equal to the number of conjugate points to $\vec{\boldsymbol{r}}(0)$ along the path $\vec{\boldsymbol{r}}(t)$ (each such conjugate point is counted with its multiplicity) [27]. In the context of vector fields on a Riemannian manifold, the index is equal to $+1$ around a source or a sink, and more generally equal to $(-1)^k$ around a saddle that has k contracting dimensions and $n-k$ expanding dimensions, and it can be shown that the sum of all of the indexes at all of the zeros must be two, because the Euler characteristic of the sphere is two. Therefore, there must be at least one zero. This is a consequence of the *Poincaré–Hopf theorem*. The theorem was proven for two dimensions by Henri Poincaré and later generalized to higher dimensions by Heinz Hopf [28]. In particular, we see how even a single free particle has a path which will be subject to some anisotropy due to the effective potential induced by the curvature of the sphere. This effect was studied in Ref. [23].

5.2. Thermal Equilibrium of the Metric Tensor

A different approach is to move the temperature from the stress–energy tensor to the metric tensor, as is done in Refs. [7,29], and also in [4–6] to study a vacuum in cosmic space in Ref. [30]. That is, to move the statistical physics description from the right-hand side of Einstein field equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{79}$$

to the left-hand side. Of course, the two descriptions must give the same picture. In Ref. [4], we used the statistical average of the trace of Einstein field equations

$$\langle -R \rangle_g = \kappa \langle T_\mu{}^\mu \rangle_t. \tag{80}$$

where $\kappa = 8\pi G/c^4$ and $R = -G_\mu{}^\mu$ is the scalar curvature, $\langle \ldots \rangle_g$ is a statistical average on the metric tensor, and $\langle \ldots \rangle_t$ is a time average. On the right-hand side, by replacing the time average with an ensemble average, we find that [21]

$$\langle z_\mu T^{\mu\nu} \rangle_t = \frac{\mathrm{tr}\left(\rho_0^{(N)} z_\mu T^{\mu\nu}\right)}{Z_N} = -\frac{\delta}{\delta \beta_\nu(\boldsymbol{x})} \ln Z_N. \tag{81}$$

In the above formula, while the left-hand side depends on an arbitrary vector $\vec{\boldsymbol{z}}$, the right-hand side is not manifestly dependent on it. In fact, the functional derivative of Z_N in Equation (76) includes a hidden dependence on the normal vector as the functional derivation implies the choice of a measure, and hence of a hypersurface and a corresponding normal vector. We will also have

$$\langle T_\mu{}^\mu \rangle_t = -\frac{\delta}{\delta \beta} \ln Z_N. \tag{82}$$

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Then, the virial theorem of Equation (80) can be rewritten as

$$\langle R\rangle_g = \kappa \frac{\delta}{\delta\beta} \ln Z_N, \tag{83}$$

where $\mathrm{tr}(\ldots)$ denotes a trace, and $\Lambda = \sqrt{4\pi\lambda\beta}$ is the de Broglie thermal wavelength, with $\lambda = \hbar^2/2m$.

For an ideal gas, where the bodies are non-interacting, $\mathscr{V} = 0$, we immediately find $Z_N = V^N/\Lambda^{-3N} N!$ where $\Lambda = \sqrt{4\pi\lambda\beta}$ is the de Broglie thermal wavelength, with $\lambda = \hbar^2/2m$. Then,

$$\langle R\rangle_g = \kappa \frac{1}{V}\frac{\partial}{\partial\beta} \ln\left[\frac{1}{N!}\left(\frac{V}{\Lambda^3}\right)^N\right] = -3n\kappa \frac{\partial}{\partial\beta}\ln\Lambda = -\frac{3n}{2\beta}\kappa, \tag{84}$$

where the functional derivative has been replaced by a partial derivative and V/Λ^3 is the single-particle translational partition function, familiar from elementary statistical mechanics.

In Ref. [4], we defined a *virial* inverse temperature $\tilde{\beta}$ stemming from the thermal fluctuations of the metric tensor (note that the path integral needed for the calculation of the left-hand side of Equation (80) in the metric tensor required the choice of Euclidean time, whereas the right-hand side requires real time) as follows:

$$\tilde{\beta}^{-1}(\boldsymbol{x}) = \frac{\tilde{v}}{4}\langle T_\mu{}^\mu\rangle_t, \tag{85}$$

where $\tilde{v}$ is a positive constant. Therefore, we find the following equivalence:

$$\tilde{\beta}^{-1}(\boldsymbol{x}) = \frac{3n\tilde{v}}{8}\beta^{-1}(\boldsymbol{x}), \tag{86}$$

or

$$\tilde{T}(\boldsymbol{x}) = T(\boldsymbol{x}), \tag{87}$$

with

$$\tilde{k}_B = \frac{3n\tilde{v}}{8}k_B, \tag{88}$$

where $n\tilde{v}$ is an intensive quantity, and $\tilde{v}$ is a local volume [4]. In Ref. [21], it was also shown that at thermodynamic equilibrium $\beta^\mu(\boldsymbol{x})$ must be a killing vector of the manifold, so it must be a constant four-vector. Then, T should be independent of $\boldsymbol{x}$ at thermodynamic equilibrium.

5.3. Thermal Equivalence Principle

This equivalence proves that Einstein field equations offer a symmetric way to study statistical physics where one can either work at the level of the many-body system encoded in the stress–energy tensor on the right-hand side of (79) or at the level of the thermal fluctuations of the metric tensor on the left-hand side of (79). The two descriptions are *equivalent*. This is a *thermal equivalence principle* in physics: "given a many-body system in general relativity its thermal equilibrium properties derived from its statistical physics description are equivalent to the properties of a statistical physics description of the metric of the spacetime that it influences and viceversa". In other words: "A statistical ensemble of bodies goes into thermal equilibrium with the spacetime it occupies". This is established by Equation (87).

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It is nice to see how the statistical physics theory and the general relativity theory can be made to be consistent one with the other. Even if this fact can be considered superfluous, in my opinion, it cannot be taken for granted or even treated carelessly and it is not at all trivial. This thermal equivalence principle issues a bridge between the two communities of statistical physics and general relativity physics.

This principle should be taken into account, for example, when constructing a more accurate and realistic equation of state of a White Dwarf, which requires us to drop the assumptions of an ideal and perfect electron gas. This has been accomplished with various mean field theories, as illustrated in section §2 of Shapiro and Teukolsky's book [9]. Since, for a White Dwarf, the Wigner–Seitz radius is small at $0.0003 < r_s < 0.01$ (the lower bound is dictated by the requirement of being below neutron drip; the upper bound can be inferred from figure 3.2 of [9], taking a star mass as low as $0.7 M_\odot$), the corrections due to the Coulomb interaction will be small. The principal effect of the electrostatic corrections is to give smaller radii and larger central densities compared with Chandrasekhar's models of the same mass. Nonetheless, in order to make further progress towards an even more accurate equation of state, many-body methods are necessary. This is not simply an academic exercise, because the stars are probably the objects most observed and measured in nature, and we can hope to better understand the laws of nature by comparing our theories on Earth with the data from astronomical observations. Until recently, stars were only present in the sky; only recently were we able to create a star artificially in a earthly laboratory [31]. Moreover, we may hope to be able to detect gravitational waves generated by binary systems of White Dwarfs as they spiral closer to an eventual merger. These systems, which can be detached or interact through mass transfer, are major sources of gravitational waves for future detectors like the Laser Interferometer Space Antenna (LISA) planned by the European Space Agency (ESA) [32]. One may then start, for example, from the properties of Jellium [33–35], where the ion component is approximated by a uniform neutralizing background. This is just a first brute approximation of the more realistic model of a two-component plasma [36,37] (note that since the mass of a proton is about 1000 electron masses, the ion component diffusion would be 1000 times slower, making it much more classical in a first-principles statistical physics description), but even so it poses the extremely challenging problem of the determination of the ground state or non-zero temperature properties of a many-electron system on a curved spacetime [23,38] with the additional subtleties of overcoming the fermion sign problem [20,39,40] and ordering problems on properly self-adjointed operators subject to holonomic constraints as the ones necessary in a quantum theory of curved spacetime [4–6,29]. The effects of temperature and/or general relativity on the global structure of a White Dwarf have been recently studied in greater depth considering realistic models of dense matter [41–43]. Moreover, full recent evolution simulations of the most massive White Dwarfs have been also carried out [44,45]. Our thermal equivalence principle could offer additional insights through the equivalent treatment of the statistical physics description of the metric of spacetime.

6. Conclusions

In this work, we determined a thermal equivalence principle for a statistical theory of gravitation. First of all, it is important to realize that at low temperatures a statistical theory of gravity will necessarily combine the quantum world with our universe, which is ruled by general relativity. This is certainly relevant for a refinement of the existing equations of state of stellar interiors, which allow the prediction and better understanding of stellar evolution. Moreover, outer space, or simply space, is the expanse that exists beyond Earth's atmosphere and between celestial bodies. It contains very low particle densities, constituting a near-perfect vacuum of predominantly hydrogen and helium plasma, permeated by electromagnetic radiation, cosmic rays, neutrinos, magnetic fields and dust. The baseline temperature of outer space, as set by the background radiation from the Big Bang, is ≈2.7 K. Intergalactic space takes up most of the volume of the universe, but even galaxies and star systems consist almost entirely of empty space. Most of the remaining mass-energy in the observable universe is made up of an unknown form, dubbed dark matter (60% of the universe) and dark energy (27% of the universe).

Constructing a well-defined statistical theory of our universe is one of the greatest challenges of contemporary physics, which had been foreseen by Einstein in his iconic phrase "God doesn't play dice". From the point of view of the challenge that it offers to mathematics, one needs a way to create a bridge between the variational theory of functional integrals, or more specifically path integrals, and differential geometry, or more specifically Riemannian geometry. From this point of view, it seems natural to predict that differential topology will play a crucial role. Recently, we carried out some path-integral (Monte Carlo) simulations for Jellium (an electron plasma at low temperature) on the surface of a sphere, probably the simplest of all curved smooth manifolds. We found important topological effects on the electron paths. It is important to realize that these kinds of calculations can be considered to be simulations for a many-body system on a more complex smooth manifold as required by spacetime in general relativity.

In this work we show that the temperature, as defined in statistical physics studies of many bodies, plays the same role as the one that can be defined by a path integral on the spacetime metric that was introduced in Ref. [4] (some may criticize [4] in that the temperature should be considered a property "external" to spacetime, but our temperature measures the thermal fluctuations of the metric tensor itself, as illustrated in Ref. [5] where the foundations of our statistical physics formulation of gravity were demonstrated). This is perfectly natural from the point of view of the linear constraint given by the Einstein field equations. This symmetry between the statistical physics of many-body matter in the universe and a statistical physics theory of the metric tensor where matter lives offers a natural thermal equivalence principle stating that the material and spacetime are in thermal equilibrium with one another. One could either let the classical many bodies live in quantum spacetime or the quantum many bodies live in classical spacetime, i.e., move the path integral description from left to right in the Einstein field equations. The two approaches have to coincide.

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