Journal of Statistical Physics (2021) 184:28
https://doi.org/10.1007/s10955-021-02818-x

Monte Carlo Evaluation of the Continuum Limit of the Two-Point Function of the Euclidean Free Real Scalar Field Subject to Affine Quantization

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Received: 10 June 2021 / Accepted: 25 August 2021

Abstract
We study canonical and affine versions of the quantized covariant Euclidean free real scalar field-theory on four dimensional lattices through the Monte Carlo method. We calculate the two-point function near the continuum limit at finite volume. Our investigation shows that affine quantization is able to give meaningful results for the two-point function for which is not available an exact analytic result and therefore numerical methods are necessary.

Keywords Monte Carlo method · Euclidean free real scalar field-theory · Canonical quantization · Affine quantization · Two-point function · Continuum limit

Mathematics Subject Classification 03.50.-z · 11.10.-z · 11.10.Gh · 11.10.Kk · 02.70.Ss · 02.70.Uu · 05.10.Ln

1 Introduction

The aim of this work is to find out what affine quantization does to a classical field-theory. The simplest such theory is a free real scalar field of mass m. In that case, the spectrum of physical states obtained with canonical quantization is known: states containing many indistinguishable particles with momenta $\mathbf{p}_1, \mathbf{p}_2, \ldots$ and energies $\sqrt{|\mathbf{p}_i|^2 + m^2}$ (here $c = 1$) obeying Bose statistics. The simplest question to ask now is: what becomes of this if the free real scalar field is subject to affine quantization [1,2] rather than canonical quantization [3]? Does the system describe particles in this case as well? If so, do they interact with one

Communicated by Luca Peliti.

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Published online: 31 August 2021

another? Working out the two-point function of the free field in that framework should be of use to answer these questions.

The free real scalar field is well understood by canonical quantization. The standard set of problems that can be resolved by canonical quantization is distinct from the standard set of problems that can be resolved by affine quantization, and one can therefore expect that an affine quantization of the classical free real scalar differs from that of canonical quantization. The purpose of this paper is to try to understand in what ways an affine quantization is similar as well as dissimilar from a canonical quantization. We add that some non-free real scalar fields have already been observed and that canonical quantization fails for several non-renormalizable fields, such as $(\phi^{12})_3$ [4] and $(\phi^4)_4$ [5]. The key to that result is the introduction of a highly unusual, additional, non-quadratic, term that is dictated by affine quantization. While affine quantization employs an additional term, that particular term formally disappears when $\hbar \to 0$, which makes it a plausible modification of the quadratic terms of traditional free real scalar fields in order to extend acceptable quantization of traditional non–renormalizable models.

The Euclidean action in canonical quantization [3], in units where $\hbar = 1$, is

$$S^{(c)}[\phi] = \int \left\{ \frac{1}{2} \sum_{\mu=0}^{s} \left[\frac{\partial \phi(x)}{\partial x_\mu} \right]^2 + V(\phi(x)) \right\} d^n x, \tag{1.1}$$

with $x = (x_0, x_1, \ldots, x_s) = (x_0, \mathbf{x})$ for s spatial dimensions and $n = s + 1$ for the number of space-time dimensions with $x_0 = ct$. We will work at $s = 3$. And V is the self-interaction potential density for which we will choose $V(\phi) = (1/2)m^2\phi^2$ corresponding to a free-theory with a bare mass m.

The Eudlidean action in affine quantization [1,2] is

$$S^{(a)}[\phi] = \int \left\{ \frac{1}{2} \sum_{\mu=0}^{s} \left[\frac{\partial \phi(x)}{\partial x_\mu} \right]^2 + \frac{3}{8} \frac{\delta^{2s}(0)}{\phi^2(x) + \epsilon} + V(\phi(x)) \right\} d^n x, \tag{1.2}$$

where $\epsilon > 0$ is a parameter used to regularize the "3/8" extra term (see Appendix A in [4]) and δ is a Dirac delta function. In this case the Hamiltonian density contains a divergent term, in the total potential density $\mathscr{V}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{3}{8}\delta^s(0)/(\phi^2 + \epsilon)$, in the continuum, but the field theory can be treated on a lattice, and the *approach* toward the continuum will be taken under exam in this work. In fact, the path integral needs this feature since we have examples such as $\int \phi^2(x) e^{-S^{(a)}[\phi]} \mathscr{D}\phi / \int e^{-S^{(a)}[\phi]} \mathscr{D}\phi$ which is a creation of $\langle \psi | \hat{\phi}^2(x) | \psi \rangle$, namely it creates a quantum version of the classical $\phi^2(x)$. The quantum operator $\hat{\phi}^2(x) \sim \delta^s(0)$ and must be passed through the functional integral which deals with terms within $S^{(a)}[\phi]$ leading to the fact that the term $\phi^2(x)$ needs to be $\sim \delta^s(0)$ (at the minima of $\mathscr{V}$) to handle the integration and that factor being "passed" to the quantum operator term $\hat{\phi}^2(x)$. In the $V \to 0$ limit, this model remains different from a massless free-theory due to exactly the new $(3/8)\delta^{2s}(0)/[\phi^2(x) + \epsilon]$ interaction term (we have a "pseudofree" situation).

In our previous works we studied the non-renormalizable canonical cases with $V(\phi) = (1/2)m^2\phi^2 + g\phi^4$ [5] in $s = 3$ and $(1/2)m^2\phi^2 + g\phi^{12}$ in $s = 2$ [4], where g is the bare coupling constant. And we showed that the corresponding affine cases are indeed renormalizable.

Monte Carlo (MC) [6,7] is the numerical method of choice to treat multidimensional integrals of high D dimensions (it supercedes the traditional integration methods, like the trapezoidal rule, the Simpson rule,. . ., based on the knowledge of the αth derivative of the integrating function already for $D > 2\alpha$) therefore is especially useful to compute path

integrals. We will use it to study the two-point function of the Euclidean action of a real scalar field in affine quantization. Our estimate of the path integrals will be generally subject to three sources of numerical uncertainties: The one due to the statistical errors, the one due to the space-time discretization, and the one due to the finite-size effects. Of these the statistical errors scale like $M^{-1/2}$ where M is the computer time, the discretization of space-time is responsible for the distance from the continuum limit (which corresponds to a lattice spacing $a \to 0$), and the finite-size effects stems from the necessity to approximate the infinite space-time system with one in a periodic box of volume L^n with $L = Na$ being the box side, subject to N discretization points.

The work is organized as follows: In Sect. 2 we derive the lattice formulation of the field theory needed in the treatment on the computer, in Sect. 3 we describe our computer experiment and introduce the observables that will be measured during our simulations, in Sect. 4 we present our results, and section 5 is for final remarks.

2 The Lattice Formulation of the Field-Theory Model

We used a lattice formulation of the field theory. The theory considers a real scalar field ϕ taking the value $\phi(x)$ on each site of a periodic, hypercubic, n-dimensional lattice of lattice spacing a and periodicity $L = Na$. The canonical action for the field, Eq. (1.1), is then approximated by

$$S^{(c)}[\phi] \approx \left\{ \frac{1}{2} \sum_{x,\mu} a^{-2} \left[\phi(x) - \phi(x + e_\mu) \right]^2 + \sum_x V(\phi(x)) \right\} a^n, \tag{2.1}$$

where e_μ is a vector of length a in the $+\mu$ direction and we are at a temperature $T = 1/Na$, in units where Boltzmann constant $k_B = 1$. An analogous expression holds for the affine action of Eq. (1.2) where the Dirac delta function is replaced by $\delta^{2s}(0) \to a^{-2s}$.

We will use this "primitive approximation" for the action even if it can be improved in several ways [8] in order to reduce the error due to the space-time discretization. In reaching to the expression (2.1) we neglected the term $\propto a^{2n}$ due to the commutator of the kinetic and potential parts of the Hamiltonian, in the Baker–Campbell–Hausdorff formula. In reaching to the path integral expression this is justified by the Trotter formula.

The affine regularization of the previous paragraphs, leading to $\mathbf{x} \to \mathbf{k}a$, where $a > 0$ is the tiny lattice spacing, is helpful in our analysis but needs not be the final regularization. In particular, the new term $\phi(x_0, \mathbf{x})^{-2} \to \phi_{\mathbf{k}}^{-2}$ leads to a divergence when, at a fixed value of $\mathbf{k}$, the integral over the region $|\phi_{\mathbf{k}}| < 1$, of $\int (\phi_{\mathbf{k}})^{-2}\, d\phi_{\mathbf{k}} = \infty$. This behavior can be overcome in an additional form of regularization.[1] Instead of just $\phi_{\mathbf{k}}$ we choose $2s$ additional terms that are nearest neighbors to $\mathbf{k}$. These additional terms enter in the form $\phi_{\mathbf{k}}^{-2} \to [\, \sum_{\mathbf{l}} J_{\mathbf{k},\mathbf{l}}\, \phi_{\mathbf{l}}^2 \,]^{-1}$, where $J_{\mathbf{k},\mathbf{l}} = (2s+1)^{-1}$ for $\mathbf{l} = \mathbf{k}$ plus $\mathbf{l}$ is each of the $2s$ nearest neighbors of $\mathbf{k}$. This averaging of $\phi_{\mathbf{k}}$ also leads to a finite integration where, with all $|\phi_{\mathbf{l}}| < 1$, we have

$$\int \cdots \int \left[\sum_{\mathbf{l}} J_{\mathbf{k},\mathbf{l}}\, \phi_{\mathbf{l}}^2 \right]^{-1} \prod_{\mathbf{l}} d\phi_{\mathbf{l}} < \infty, \tag{2.2}$$

which is finite as determined by choosing $\phi_{\mathbf{l}} = r\, u_{\mathbf{l}}$ such that $\sum_l u_{\mathbf{l}}^2 < \infty$ leading to the integral $U \int r^{-2} r^{2s} dr < \infty$, for all $s > 0$, where $U < \infty$ accounts for the remaining finite integrations.

[1] The additional regularization is essentially taken from Eq. (14) in [9].

Clearly, this procedure of averaging the expression $\phi_{\mathbf{k}}^{-2}$ offers a smoother regulation, and we shall also adopt that procedure for our MC studies. We will refer to this affine regularization as term B and the one discussed earlier, obtained by choosing $J_{\mathbf{k},\mathbf{l}} = \delta_{\mathbf{k},\mathbf{l}}$, as term A.

The vacuum expectation of a functional observable $\mathscr{O}[\phi]$ is

$$\langle \mathscr{O} \rangle \approx \frac{\int \mathscr{O}[\phi] \exp(-S[\phi]) \prod_x d\phi(x)}{\int \exp(-S[\phi]) \prod_x d\phi(x)}, \tag{2.3}$$

for a given action S.

We will approach the continuum limit by choosing a fixed L and increasing the number of discretizations N of each component of the space-time. So that the lattice spacing $a = L/N \to 0$. To make contact with the continuum limit, two conditions must be met $a \ll 1/m \ll L$ where $1/m$ is the Compton wavelength.

3 Simulation Details and Relevant Observables

We want to determine the two-point function

$$K(x, y) = \langle [\phi(x) - \langle \phi(x) \rangle][\phi(y) - \langle \phi(y) \rangle] \rangle = \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle^2, \tag{3.1}$$

replacing x by $x + k$ with $k = aw_n$ with $w_n = (n_0, n_1, \ldots, n_s)$ and $n_\mu \in \mathbb{Z}$ amounts to a mere relabeling of the lattice points. Hence, due to translational invariance, $K(x, y)$ can only depend on the difference between the coordinates of the two points and we can define,

$$D(z) = \frac{1}{L^n} \sum_x K(x, x + z) a^n, \tag{3.2}$$

For the massless free-theory with $V \to 0$ in canonical quantization, we find that in non periodic space-time (at zero temperature)

$$D'(z) = \int \frac{e^{-ip\cdot z}}{p^2} \frac{d^n p}{(2\pi)^n} = \begin{cases} -|z|/2 & n = 1 \\ -(\ln |z|/l)/2\pi & n = 2 \\ 1/|z|4\pi & n = 3 \\ 1/|z|^2 4\pi^2 & n = 4 \end{cases}, \tag{3.3}$$

where $|z| = \sqrt{z_0^2 + z_1^2 + \cdots + z_s^2}$ and l is a length. This shows how the massless field generates long range interactions.

For a massive free-theory with $V(\phi(x)) = \frac{1}{2} m^2 \phi^2(x)$ in canonical quantization, we find that in non periodic space-time (at zero temperature) with n = 4

$$D'(z) = \int \frac{e^{-ip\cdot z}}{p^2 + m^2} \frac{d^n p}{(2\pi)^n} = m \mathrm{K}_1(m|z|)/|z|4\pi^2, \tag{3.4}$$

where m is the mass and K_1 is a modified Bessel function.

In periodic space–time (at a temperature $T = 1/Na$)

$$D(z) = \sum_{w_n} D'(z + Lw_n), \tag{3.5}$$

where the sum can be restricted by an infrared cutoff irc such that $-irc \le n_\mu \le irc$ (without any physical significance) in order to reach a given numerical accuracy. If we remove the cutoff the function diverges for the massless case.

Our MC simulations use the Metropolis algorithm [6,7] to calculate the ensemble average of Eq. (2.3) which is a N^n multidimensional integral. The simulation is started from the initial condition $\phi = 0$. One MC step consisted in a random displacement of each one of the N^n $\phi(x)$ as follows

$$\phi \to \phi + (2\eta - 1)\delta, \tag{3.6}$$

where η is a uniform pseudo random number in [0, 1] and δ is the amplitude of the displacement. Each one of these N^n moves is accepted if $\exp(-\Delta S) > \eta$ where ΔS is the change in the action due to the move (it can be efficiently calculated considering how the kinetic part and the potential part change by the displacement of a single $\phi(x)$) and rejected otherwise. The amplitude δ is chosen in such a way to have acceptance ratios as close as possible to $1/2$ and is kept constant during the evolution of the simulation. One simulation consisted of M MC steps. The statistical error on the average $\langle \mathscr{O} \rangle$ will then depend on the correlation time necessary to decorrelate the property $\mathscr{O}$, $\tau_{\mathscr{O}}$, and will be determined as $\sqrt{\tau_{\mathscr{O}} \sigma^2_{\mathscr{O}}/(MN^n)}$, where $\sigma^2_{\mathscr{O}}$ is the intrinsic variance for $\mathscr{O}$.

4 Simulation Results

We worked in units where $c = \hbar = k_B = 1$. We chose the regularization parameter of the affine quantization A term to be $\epsilon = 10^{-10}$.[2]

For a massive free-theory, $V(\phi) = \frac{1}{2}m^2\phi^2$, in canonical quantization (1.1) with $m = 1, N = 15, L = 3, a = L/N = 0.2$ we obtained the result shown in Fig. 1 where we compare the MC results with the exact expression of Eq. (3.5) with an infrared cutoff of $irc = 2$ which is sufficient for an accuracy of 10^{-3}. The run was $M = 10^6$ MC steps long. The figure shows good agreement between the MC and the exact expression except at the origin due to the space-time discretization.

For a free massive theory $V(\phi) = \frac{1}{2}m^2\phi^2$ in affine quantization (1.2) using term A, the self-interaction is a double well with a spike barrier at $\phi = 0$. We tuned the width of the displacement, δ in Eq. (3.6), so that the random walk in the $\phi(x)$ will sample the probability distribution $\exp(-S[\phi])$ most efficiently, with short equilibration times. In Fig. 2 we show the result for a free real scalar field subject to affine quantization with a total self-interaction of the form $\mathscr{V}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{3}{8}a^{-2s}/(\phi^2 + \epsilon)$ with $m = 1$, $N = 15$, $L = 3$, $a = L/N = 0.2$, and $\epsilon = 10^{-10}$ after cutting the first equilibration MC steps of a run made of $M = 2.5 \times 10^6$ steps. During the simulations we also calculated the renormalized mass m_R and the renormalized coupling constant g_R [4]. As we can see from the figure the symmetry $z \to L - z$ of the two-point function is preserved within the errorbars. The minima of the classical $\mathscr{V}$ is at $\phi = \pm\Phi$ with $\Phi^2 = -\epsilon + \sqrt{3}/(2a^3m)$ which diverges in the continuum limit $a \to 0$ (this of course does not happen in the harmonic oscillator case [10] which is independent of the lattice spacing). Moreover the minimum of the action $L^{s+1}m(\sqrt{3} - m\epsilon a^s)/2a^s$ also diverges, both in the continuum limit at finite volume ($ma \to 0$) and in the infinite volume limit at fixed lattice spacing ($mL \to \infty$) (this also happen for the affine harmonic oscillator [10] which has a well defined zero temperature limit). The corresponding contribution to the vacuum expectation only occurs together with the normalization constant in front of the path integral and drops out in quantities of physical interest (as long as the system is not placed in

[2] Note that we could as well choose a regularization putting hard walls at $\phi = \pm\varepsilon$ therefore rejecting MC moves whenever $\phi \in [-\varepsilon, \varepsilon]$

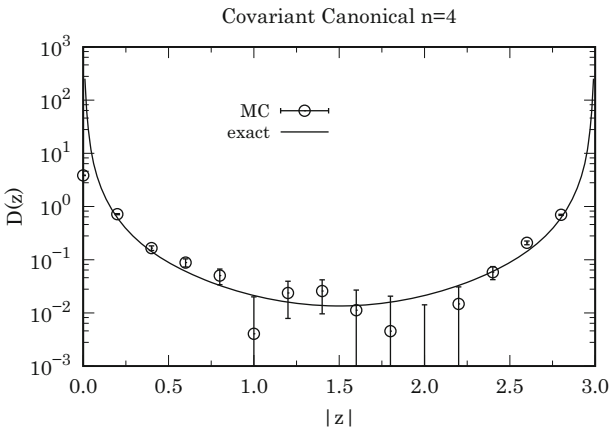

Fig. 1 Two-point function $D(z)$ of Eq. (3.2), for a free real scalar field subject to canonical quantization with a self-interaction potential density of the form $V(\phi) = \frac{1}{2}m^2\phi^2$ in Eq. (1.1) with $m = 1$, $N = 15$, $L = 3$, $a = L/N = 0.2$. We compare with the analytic exact expression of Eq. (3.5) with an infrared cutoff of $irc = 2$. A logarithmic scale is used on the y-axis

a curved geometry, i.e. in a gravitational field - there, the cosmological constant does have physical significance)

The symmetry $\phi \to -\phi$ is broken in the simulations (see Appendix 1) and as a result $\langle\phi(x)\rangle$ is different from zero. The action $S = \bar{K} + \bar{V}$ where $\bar{K}$ is the kinetic term and $\bar{V}$ the total potential term. Imagine now that we are in a configuration where all the N^n components, $\phi(x)$, are around $+\Phi$. In order to start migrating one single x' component, $\phi(x')$, around the other minimum at $-\Phi$ will have no cost in the potential, $\Delta\bar{V} \approx 0$, but it will have a big cost in the kinetic term between "neighboring" x, resulting in a big $\Delta\bar{K}$ (as long as the distance between the two minima, 2Φ, which diverges in the continuum limit, is large). As a consequence $\exp(-\Delta S)$ will be very small and the move will be almost surely rejected according to the Metropolis rule. Moreover, once the system reaches the phase with all $\phi(x)$ in one of the minima, it is very unlikely that a single $\phi(x')$ will move to the other minimum but it cannot be excluded, in principle. If this happens one has a situation where the field is around $+\Phi$ at all x except at x' where it is around $-\Phi$. But we can easily see that now it would be statistically favorable for the single field on the left to rejoin the fields on the right other then all the fields on the right join the field on the left. Exactly the same holds for affine quantization (1.2) using term B, since due to the kinetic energy term in the action the fields at neighboring points tend to assume similar values. On the other hand this would not hold for an *ultralocal* [11] theory where we could have the field visiting both wells at $\pm\Phi$ but only at not "neighboring" times, resulting in a vanishing $\langle\phi(x)\rangle$. Apart from this the shape of the two-point function is qualitatively similar to the one of the *covariant* case of Eq. (1.2). In addition in a covariant *complex* field one could go "slowly" "around" the "mountain" at $\phi = 0$ with no need of "jumps".

For our choice of the parameters we have $\Phi \approx 10.404$ with $\Phi^2 \approx 108.253$. The results in Fig. 2 indicate that the quantization increases this number by about 10%. The minimum of $D(z)$ is reached around $|z| = L/2$. The two-point function is qualitatively similar to the one of the free field. This is supported by recent results on a one dimensional harmonic oscillator treated with affine quantization [10] where it is shown that the eigenvalues are still equally spaced. A non-linear fit of the MC data (removing the first point at $|z| = 0$) with the function

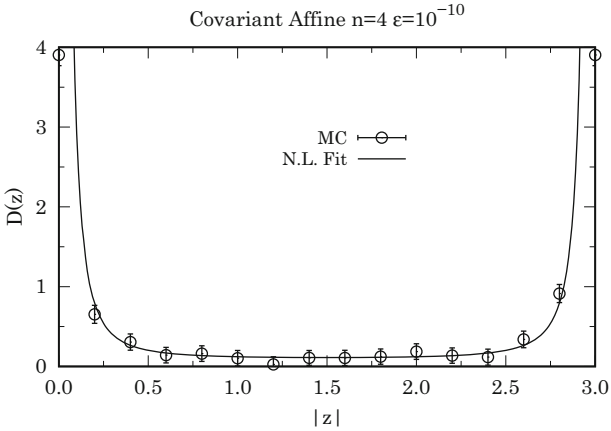

Fig. 2 Two-point function $D(z)$ of Eq. (3.2), for a free real scalar field subject to affine quantization with term A and a self-interaction potential density of the form $V(\phi) = \frac{1}{2}m^2\phi^2$ in Eq. (1.2) with $m = 1$, $N = 15$, $L = 3$, $a = L/N = 0.2$, and $\epsilon = 10^{-10}$. Also shown is the result of a non-linear fit of the data (except the first point at $|z| = 0$) with the function $D_{m_D}(z)$ where D_{m_D} is the two-point function of a free field of mass m_D of Eq. (3.5) with an $irc = 2$, taking m_D as the only fit parameter

$D_{m_D}(z)$ where D_{m_D} is the two-point function of a free field of mass m_D of Eq. (3.5) with an $irc = 2$, taking m_D as the only fit parameter, gives $m_D \approx 0.9$. The result of the fit is also shown in Fig. 2.

For a free real scalar field subject to affine quantization with term A, in $n = 4$ space-time dimensions in a volume 3^4 with a regularization parameter $\epsilon = 10^{-10}$, we studied the continuum limit, $N \to \infty$, (by choosing values lower of 15) and the dependence on the bare mass m, of the five quantities m_R, g_R, $\langle\phi(x)\rangle^2$, m_D, and $D(0)$. The results are shown in Table 1. From the table we see how moving towards the continuum limit $m_D \approx m$ but m_R becomes small due to the fact that when the field picks up an expectation value, the Fourier transform of the field $\widetilde{\phi}(0)$ picks up a contribution proportional to the volume of the box. Moreover, for the same reason, $g_R \approx 2$. The Table also shows the value of Φ^2 and of $\langle\phi(x)\rangle^2$ to be compared. We see that the second is always larger than the first one by a percentage increasing with increasing m and with increasing a. The value of $D(0)$ is increasing with a decrease of the lattice spacing a, signaling a divergence in the continuum limit.

Summarizing, the two-point function for $\phi - \langle\phi\rangle$ looks similar to the two-point function of a free field with mass m_D. In other words, the correlation length of the affine quantum field theory is m/m_D times the Compton wavelength of the canonical quantum theory of the free scalar field. Our results seem to suggest that, going towards the continuum, the affine model is approaching a free field with the same bare mass.

The value of m_D is not easy to understand, however. If the action is treated at the classical level, small deviations from the minimum are determined by the curvature of the total potential, $m_c^2 = d^2\mathscr{V}/d\phi^2$ at $\phi = \Phi$. The mass term contributes m^2 and the "3/8" term yields a contribution that is 3 times larger. For $\epsilon = 0$, the mass relevant for the relation between frequency of the waves and wavelength is: $m_c = 2m$ independently of a.

In Fig. 3 we show $D(z)$ as obtained for $m = 1$ ($L = 3$, $\epsilon = 10^{-10}$) and three choices of N, in the long simulations of the Table 1. One can then see the approach to the continuum of the two-point function of the affine model.

Table 1 We determined, for a free real scalar field subject to affine quantization with term A, in $n = 4$ space–time dimensions, the dependence of m_R, g_R, m_D, and $D(0)$ on the number of one dimensional discretization points N and the bare mass m at $L = 3$ with a regularization parameter $\epsilon = 10^{-10}$

N	m	m_R	g_R	Φ^2	$\langle\phi(x)\rangle^2$	m_D	$D(0)$
15	1	0.0122(3)	1.9979(1)	108.2	120.6(1)	0.934	3.69(6)
	2	0.00646(4)	1.99983(3)	54.13	65.7(1)	1.785	3.32(6)
	3	0.0186(6)	1.99925(8)	36.08	45.85(7)	3.009	2.97(6)
12	1	0.01053(5)	1.99958(5)	55.43	63.25(8)	0.302	2.38(5)
	2	0.00967(9)	1.99992(2)	27.71	34.54(5)	2.467	2.00(5)
	3	0.0095(1)	1.99905(8)	18.47	24.00(4)	5.483	1.66(5)
10	1	0.01417(4)	1.999464(4)	32.07	37.46(5)	0.587	1.58(3)
	2	0.0124(1)	1.99995(1)	16.04	20.43(3)	3.789	1.29(3)
	3	0.0119(2)	1.99996(1)	10.69	14.03(2)	5.647	1.02(3)

The runs were $M = 5 \times 10^6$ MC steps long. The value of Φ^2 and of $\langle\phi(x)\rangle^2$ are also shown for comparison

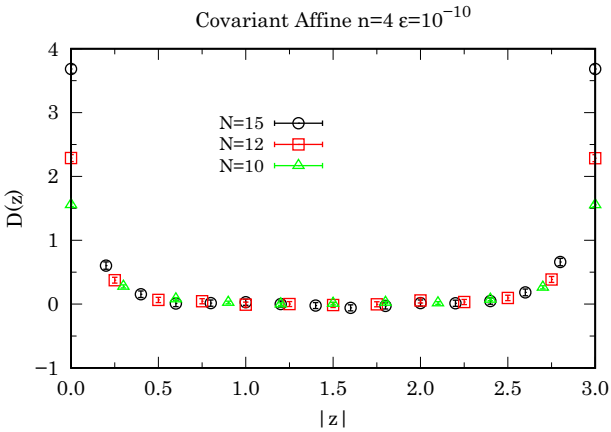

Fig. 3 (color online) Two-point function $D(z)$ of Eq. (3.2) for a free real scalar field subject to affine quantization with term A and a self-interaction potential density of the form $V(\phi) = \frac{1}{2}m^2\phi^2$ in Eq. (1.2) with $m = 1$, $L = 3$, $\epsilon = 10^{-10}$ and increasing $N = 10, 12, 15$

5 Conclusions

In a recent work [5] we studied the case of a non-renormalizable $(\phi^4)_4$ canonical theory (where the self-interaction potential is $V(\phi) = g\phi^4$) in four space-time dimensions and proved through MC that the theory becomes renormalizable if one treats the field through affine quantization.

In the present work we observed that for $g = 0$ the simplest question to ask was: Does the affine system describe particles as for the canonical one? If so, do they interact with one another?

We tried to answer these question by looking at the two-point function. What we proved through our MC analysis was that the affine case with $g = 0$ has to be considered like a "sort" of free-theory of "quasiparticles" (in the sense of Lev Landau in his theory for Fermi liquids) where the "3/8" term just offers itself like a sort of "collective excitation" term. In this case

the $\phi \to -\phi$ symmetry is broken and the field acquires a non-zero vacuum expectation. The two-point function nonetheless has all the same features as those of a free scalar field of similar mass, in the continuum limit.

One shortcoming of the affine formulation of the field theory is the divergence (in the continuum) of the vacuum expectation value of the field which generates the disconnected contribution to the Green's functions. The path integral is fully determined by the local properties of the field that enter through the action. The expectation value of the field does not represent a local property of the field. We cannot imagine how one could possibly get rid of it. In the Standard Model, however, one of the crucial properties of the Higgs fields is that they pick up a vacuum expectation value v. The masses of the W- and Z-bosons as well as those of the leptons and quarks are proportional to v. In order to remedy to this drawback one should perform the following scaling $\phi \to a^{-s/2}\phi$ (together with $g \to a^s g$ in a possible interaction term of the form $g\phi^4$) which would bring about an additional factor a^{-s} multiplying the action. This scaling proved successful in our forthcoming work on the affine quantization of a Higgs complex scalar field [12].

The present paper is wanted to confirm that both canonical and affine procedures lead to desired and expected behavior for quadratic potential terms. A later paper [12] will be designed to deal with quartic potential terms with canonical and affine procedures.

Acknowledgements Many thanks to Heinrich Leutwyler for his suggestions, comments, and someone who proposed the canonical and affine programs with regard to the required features including what to examine and what to expect, which has influenced our program and its results, and led to many positive results and highlighted the expected canonical and affine differentiation.

Appendix: Field Configurations in the Vicinity of the Two Degenerate Minima in the Affine Version

Classically, the affine version of the free Hamiltonian has two degenerate minima, $\phi = \pm\Phi$. If the path integral is dominated by those field configurations that are located in the vicinity of one of these everywhere on the entire lattice or in the vicinity of the other, then it consists of two equal pieces

$$Z = \int \mathscr{D}\phi \, \exp(-S[\phi]),$$

$$Z_+ = \int \mathscr{D}\phi \, \exp(-S[\phi]), \quad \text{integral only over } \phi(x) \approx \Phi,$$

$$Z_- = \int \mathscr{D}\phi \, \exp(-S[\phi]), \quad \text{integral only over } \phi(x) \approx -\Phi,$$

and $Z_+ = Z_-$. Under a broken symmetry $\phi \to -\phi$ one would get either $Z \approx Z_+$ or $Z \approx Z_-$. This has to be expected in the present case of a real field since in order to move the field $\phi(x)$ at a single x from around Φ to around $-\Phi$ in the MC path integral one has to overcome a large kinetic cost. This is not true for a complex field where one can go "slowly" "around" the "mountain" at $\phi = 0$.

The expectation value of the field

$$\langle \phi(x) \rangle = \int \mathscr{D}\phi \, \phi(x) \exp(-S[\phi])/Z,$$

$$\langle \phi(x) \rangle_+ = \int \mathscr{D}\phi \, \phi(x) \exp(-S[\phi])/Z_+, \quad \text{over } \phi(x) \approx \Phi$$

$$\langle\phi(x)\rangle_- = \int \mathscr{D}\phi\, \phi(x) \exp(-S[\phi])/Z_-, \quad \text{over} \;\; \phi(x) \approx -\Phi,$$

with $\langle\phi(x)\rangle_+ \approx \Phi$, $\langle\phi(x)\rangle_- \approx -\Phi$, and under the broken symmetry, $\langle\phi(x)\rangle \approx \langle\phi(x)\rangle_\pm \approx \pm\Phi$ where the simulation, starting from $\phi = 0$, will choose among the two different cases just after the first equilibration steps.

For the two-point function

$$D_+(x-y) = \int \mathscr{D}\phi\, \phi(x)\phi(y) \exp(-S[\phi])/Z_+ - \langle\phi(x)\rangle_+^2, \quad \text{over} \;\; \phi(x) \approx \Phi,$$

$$D_-(x-y) = \int \mathscr{D}\phi\, \phi(x)\phi(y) \exp(-S[\phi])/Z_+ - \langle\phi(x)\rangle_-^2, \quad \text{over} \;\; \phi(x) \approx -\Phi,$$

so that $D_+(z) \approx 0$, $D_-(z) \approx 0$, and $D(z) \approx D_\pm(z) \approx 0$.

Moreover one can see how in the broken symmetry configuration in which $\phi^2(x) \approx \Phi^2 \sim a^{-3}$, the "3/8" term in the Hamiltonian density is also of the same order in the continuum limit $a \to 0$. This will lead to a convergent two-point function for $\phi - \langle\phi\rangle$ in the continuum limit.

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